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MATHEMATICS

FOR THE

ACCOUNTANT

BY

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PREFACE

This book is the outgrowth of a course which has been given for several years in the School of Commerce and Finance of Northeastern College. It has always been the policy of the School to give its students a thorough training in all phases of accountancy, and annuity studies have been considered a part of the training for that profession. While the course is forming in the early part of the year, it has seemed well to treat some elementary subjects which frequently appear in C. P. A. examinations, such as averaging of accounts and foreign exchange, and at the end of the year it has seemed practical to include a few lectures on the slide rule

A new impetus was given to tne work by an editorial which appeared in the "Journal of Accountancy" for August, 1918. By that editorial the American Institute of Accountants committed itself definitely to the policy of requiring certain annuity studies of all candidates for its examinations. The editorial read in part as follows: "The scope of the examination in Actuarial Science is to include certain problems relative to interest and annuities, certain sinking funds, loans repayable by instalments, and so forth, and also the construction and use of tables relating thereto. In other words, the candidate will be expected to answer questions based upon a knowledge which will have been obtained in the study of algebra.

Any candidate who has an intelligent conception of algebra should have no difficulty in answering the questions which will be propounded in actuarial science. . . .

"There is a great deal to be said in favor of the inclusion of actuarial problems and we believe that the need for knowledge of this kind will increase as time goes on. Heretofore there have been many accountants who have had practically no need to exercise any knowledge which they have possessed of actuarial matters, but with the growth of accounting work and the broadening of its scope, there must be many problems which can be solved at a great saving of time and effort if the accountant be able to deal with them with the advantage of actuarial knowledge."

This requirement seems to imply a general study of compound interest and its application to those problems which are commonly solved by simple interest. It is axiomatic that every dollar in the business world is at work; it is earning other dollars either for its owner or for someone else. At the end of every fiscal period all these earnings, after deduction of expenses, should become capital, and the same or a larger income rate should be earned on this increased capital. Such studies are compound interest and present worth, with the various aspects of interest rates; annuities, immediate, due or deferred; their amount and present worth; sinking funds and various related problems in valuation of assets; amortization; bond valuation in its many aspects.

It is obvious that the accountant has no time for algebraic studies, unless he has attended a good high school or college. To prepare a book for those who have not had these advantages and who nevertheless are promising students of accountancy, all the work must be reduced to a basis of arithmetic and common sense. Formulae there must be, and they are numerous and sometimes complicated; yet there is no reason why any man who is fitted mentally for high grade accounting practise should have difficulty with any formulae which are germane to his profession. The interest rates are so varied and so seldom common enough to be included in annuity tables that a study of logarithms is absolutely necessary. This study should be pursued no further than is demanded by the formulae for compound amount and the simpler formulae for the time and rate involved in a transaction. In short, the difficulty has been to take several highly technical books on actuarial science and reduce them to a content and method which will give the accountant a maximum of training in a minimum of time. Some actuarial works are suitable for the student who knows some algebra and likes to read mathematics. Such are Todhunter's "Textbook of the Institute of Actuaries, Book I", King's "Theory of Finance", and Mackenzie's "Interest and Bond Values". For the student who knew algebra once and desires to renew his knowledge there are the Wentworth-Smith "Commercial Algebra, Book 2", and Skinner's "Mathematical Theory of Investment". For the student who has no algebra there is only the Sprague-Perrine "Accountancy of Investment", which is written from the standpoint of the savings bank man or trustee, and does not treat some important phases of general accounting knowledge. These are all excellent books, and have been of great service to the author.

The most convenient tables for the student are those in Skinner's "Mathematical Theory of Investment", and reprinted separately by the publishers, Ginn and Company.

This book, then, is for the use of the accountant who desires to prepare for the examinations of the American Institute of Accountants or for the Certified Public Accountants' examinations of the various states; for the accountant who will be required to handle sinking fund or bond accounts in a scientific way; for the accountant who may be called upon to audit the accounts of an insurance company, savings bank, brokerage house, or any concern which operates its bond accounts scientifically; and finally for the accountant who believes that the broadest training is best for the professional man.

The book ends with a chapter in which the fundamentals of actuarial science are treated in a different manner, as a quiz for candidates for examinations. In this chapter the treatment is wholly arithmetical, and no logarithmic or algebraic knowledge is required. Problems from the recent examinations of the American Institute of Accountants and from some recent C. P. A. examinations are solved and discussed. Since no books or tables are allowed in any of these examinations, the rules are presented in such form as makes them easy to remember and apply.

Inasmuch as this book is written for the benefit of accountants, their co-operation is requested and will be gratefully accepted. Any suggestions as to content, method, or problems will be acknowledged and considered seriously. Any formulae which accountants have found useful are particularly welcome. It is planned that if the book goes through a revision a collection of formulae shall appear as a separate chapter, a reference list such as engineers find so useful. Any C. P. A. problems will also be especially welcome, because the aspects of the subject which appear important to examiners are so varied that the most complete collection which can be put together will be none too complete for the prospective candidate to study.

In closing, I wish to express my sincere appreciation of the help and inspiration I have received from Professor Charles F. Rittenhouse, C. P. A. He first brought the subject to my attention as suitable matter for a college course, and has in many ways inspired the writing of this book.

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CHAPTER I

PRELIMINARY SUGGESTIONS

This chapter is a collection of processes which will prove to be timesavers in the use of tables. It is not in any sense a collection of short cuts such as is found in the texts on arithmetic, but simply a brief consideration of methods of shortening and simplifying the numerous and otherwise laborious operations which are necessary when tables of annuities or logarithms have to be used extensively. It is obvious that if an operation can be performed one way and lead to a ten-place answer, and by another method will give only a four-place answer, that the second method is preferable for the accountant. These processes are called reversed or contracted multiplication, and contracted division. A brief consideration of the number of decimal places desirable in any given computation will also be found helpful.

Table of Multiples: In any case where the same number must be used often as multiplicand or divisor, a table of multiples is a time saver. It is formed as follows: Suppose we are finding interest at exact time; that is we are using 365 as a divisor:

1	365
2	730
3	1095
4	1460
5	1825
6	2190
7	2555
8	2920
9	3285
10	3650

On the first line set down the number. The next line is twice the number. The third line is the sum of the first and second. The fourth line is the sum of the first and third; and so on. The tenth, which is the sum of the first and ninth, is also ten times the original number, and so is a proof line. Now in multiplying or dividing we can read the desired multiples and quotient figures from this table

Contracted multiplication, sometimes called reversed multiplication. Suppose we are to receive \$459.73 at the end of five years, and we wish to "discount" this amount to the present time, at 5% compounding annually. \$1.00 due in five years at 5% compounding annually is worth \$.7835262 now; this is called the present worth. So the present worth of \$459.73 is the product of these amounts. Form a table of multiples:

1	45973
2	91946
3	137919
4	183892
5	229865
6	275838
7	321811
8	367784
9	413757
10	459730

The product will run to nine places of decimals and we need only four. We can do away with the five superfluous digits by beginning with the left-hand digit of the multiplier and proceeding as follows:

```
  459.73
 .7835262
 --------
 321.811
   36.7784
    1.3792
      2299
        91
        27
 ---------
 360.2103
```

Locate the decimal point in the first partial product. This is usually a matter of common sense, rather than rule. In this example, $450 \times .8 = 360$; that is, there are three digits at the left of the point.

In the second partial product we have four places and are ready to reject all digits at the right of the fourth. But it is advisable to take account of all "carrying" digits. So the third partial product ought to be 13792 rather than 13791. In all such multiplication as this the right-hand figure is inaccurate, but the error is negligible if we are keeping a sufficient number of decimal places.

Contracted division: The above problem could have been solved by division. The compound amount of $1.00 for five years at 5% compounding annually is $1.2762816. Therefore the number of dollars which will amount to 459.73 will be the quotient of 459.73 divided by 1.2762816. We will not show the table of multiples here.

```
12762816)4597300000(360.21047
         38288448
         --------
          76845520
          76576896
          --------
            2686240
            2552563
            -------
             133677
             127628
             ------
               6049
               5105
               ----
                944
                893
                ---
```

The division proceeds as usual until we reach the decimal point, after which we do not bring down any more digits from the dividend, but at each operation cut off the right hand digit of the divisor.

The number of decimal places necessary for accuracy varies. In general, the more the better; for the accuracy of any result is somewhat less than the accuracy of the least accurate item.

In ordinary dollars and cents work, as in schedules of annuities and bonds, all values ought to be carried to four places, that is to hundredths of a cent.

In logarithmic work, there ought to be not less than six places, and seven are better. Logarithms of the compound interest ratios must run to at least ten places, because the rates are so close together and so near the beginning of the number scale that less than ten places will not show the differences accurately. Furthermore there are so many cases in which the results of these operations are multiplied by 100,000 or more, that less than ten digits will not give even approximately correct results. Even then it is best to test out all results by reversing the operation or setting up a schedule, then if an error appears it can be spread over the entire life of the transaction by methods which will be shown later.

PROBLEMS ON CHAPTER I

1. Required the present worth of $2,283.67 due in 10 years at 4% compounding annually:
   (a) Solve by multiplying by $.6755642, the present worth of $1.00 due in 10 years at 4% annually.
   (b) Solve by dividing by $1.4802443, the compound amount of $1.00 for the same time and rate.
2. Out of 89,032 persons living at age 25, 74,173 are alive at age 45. Find by division the probability that a person now aged 25 will live to age 45.
3. An asset costing $750.00 has an estimated life of five years and scrap value of $100.00. By one method of reckoning depreciation it is estimated that this asset depreciates each year 33.17% (or as a decimal .3317) of its value at the beginning of that year. Construct a schedule showing the net value of the asset year by year, as follows:

Cost	$750.00
Less depreciation, first year—	
750 × .3317	248.775
Value, beginning of second year	$501.225
Less depreciation—501.225 × .3317	166.2563

And so on.

Value at end of fifth year should be almost exactly $100.00.

4. A Profit and Loss Statement shows:

Net Sales.		$167,834.29
Cost of Sales.	$82,387.49	
Selling Expenses.	31,093.77	
G. & A. Expenses.	22,798.44	
Net Profit.	xx,xxx.xx	

What per cent of the Net Sales is each of these items? Total of per cents should of course be 100%. (Form a table of multiples of the divisor; why?)

CHAPTER II

SIMPLE INTEREST

"Interest is the increase of indebtedness through the lapse of time."—Sprague.

Any interest contract must take account of four things: **Principal,** the number of units originally invested. **Rate,** the part of a unit of value, usually a few hundredths, which is added to each unit of principal by the lapse of one period of time. **Frequency,** the length of a unit of time, measured in years, months, or days; weeks are not used, nor are parts of days, **Time,** the number of these units of time during which the indebtedness continues.

Some of the above are usually represented by letters. The principal is P, the rate is i, the time is n. More common than i is the symbol for the amount at the end of one period, $1 + i$; this symbol is constantly used in investment mathematics. For instance, on one dollar for one year at 3% $1 + i$ is 1.03, which means that at the end of each period the debt has increased to 1.03 of its size at the beginning of the period.

Since each dollar increases at the same rate as every other dollar, it is correct and usual to reckon the interest or amount of one dollar, and multiply the result by the number of dollars stated in the problem.

There are three common methods of reckoning simple interest. They differ according to the way in which the time is reckoned.

By common time the year is divided into 12 months of 30 days each, regardless of the calendar. This method is unsatisfactory, and ought never to be used by any person who claims to aim at accuracy.

By exact time the year is divided into 365 days, and no account is made of months. This is the just method of reckoning interest, and is simple enough, especially when exact interest tables can be used.

By bankers' time the methods are combined. The exact number of days is computed, and the result is divided into groups which are multiples or fractions of 60 days. If a bill is timed three months from June 19 it is due September 19; but if it is timed 90 days from June 19 it is due September 17. There are many variants of this method, and only one of them will be treated here.

Since most interest calculations are for short periods, the results obtained under the various methods do not differ greatly. Moreover, it is a simple matter to change from common to exact time or from exact time to common time. If we let I represent the interest at common time and I′ the interest at exact time, we have two equations:

To change from common to exact $I' = I - \frac{1}{73.}$

Rule: From the ordinary interest subtract 1/73 of itself.

To change from exact to common $I = I' + \frac{1}{72}$

Rule: To the exact interest add 1/72 of itself.

Exact interest: To find the exact interest on $382.00 at 5% from May 11 to October 1. First count the days, 143. Then reckon one year's interest, $19.10. We have now to divide 19.10 by 365 and multiply by 143; but it is easier and more accurate to multiply first.

One year's interest..............	19.10
times 143......................	2,731.30
divided by 365..................	7.48

In such computation a table of the multiples of 365 is useful, and also a table of the time between dates, or a calendar showing the number of each day in the year.

One percent method: Interest by bankers' time is usually figured by some variation of the one percent method; when the rate is 6% it is usually called the 60-day or 200-month method.

At 6% one year's interest on $1.00 is six cents; therefore an interest of one cent will be earned in 1/6 of a year, which by this method is 60 days. In general, any principal at 6% will earn 1% of itself in 60 days. And since 6 days is 1/10 of 60 days, any principal will earn 1/10 of 1% of itself in six days.

Rule: At 6%, pointing off two places in the principal gives the interest for 60 days; pointing off three places gives the interest for six days.

Any period of time can be divided into multiples and fractions of 60 days, as follows:

120 days is twice 60 days.
180 days is 3 times 60 days, and so on.
30 days is 1/2 of 60 days.
20 days is 1/3 of 60 days.
15 days is 1/4 of 60 days.
12 days is 1/5 of 60 days.
10 days is 1/6 of 60 days.
6 days is 1/10 of 60 days.
3 days is 1/2 of 6 days.
2 days is 1/3 of 6 days.
1 day is 1/6 of 6 days.

It is often convenient to build in other ways:

15 days is 1/2 of 30 days.
2 days is 1/10 of 20 days, and so on.

Problem: Find the interest on $125.00 from April 18 to August 3 at 6%, bankers' time.

The number of days is:

April	12 days
May	31 days
June	30 days
July	31 days
August	3 days
Total	107 days

Divide the time as follows:

60	days'	interest	$1.25	(point off 2 places)
30	"	"	.625	(1/2 of 60)
10	"	"	.2083	(1/3 of 30)
6	"	"	.125	(1/10 of 60)
1	"	"	.0208	(1/10 of 10)
Total			$2.2291	
or, rounding up			2.23	

If the rate is not 6% it is usually easier to find at 6% and adjust. For instance, to find at 5%, after finding at 6% divide by 6 and subtract. It is possible, however, to work out special methods for some rates:

At 5%,	pointing off 2 places	gives	interest	for	72	days
At 4%,	"	"	"	"	90	"
At 4½%,	"	"	"	"	80	"
At 3%,	"	"	"	"	120	"

No such device can be used for odd rates, such as 6½%, 4½%, and so forth.

Problem: What is 119 days' interest on $147.35 at 5%, bankers' time?

72 days' interest		$1.4735	(point off 2 places)
24	" "	.4913	(1/3 of 72)
18	" "	.3683	(1/4 of 72)
4	" "	.0818	(1/6 of 24)
1	" "	.0204	(1/4 of 4)
119		$2.4353	
	or	$2.44	

Discounting notes: Find the net proceeds of a 90-day note for $350.00, dated May 19, interest at 5%, discounted June 2 at 6%.

Face....................................	$350.00
Add 90 days' interest at 5%............	4.38
Value at maturity.......................	$354.38
Due date, 90 days from May 19 is Aug. 17	
Term of discount, from June 2 to Aug. 17 is 76 days	
Discount, 76 days at 6% on $354.38....	4.49
Net proceeds...........................	$349.87

Interest on daily balances: Many banks advertise interest on daily balances of checking accounts at a nominal rate. One method of computing this interest is the following: Each day's balance is made out, and at the end of the month these balances are totalled and one day's interest is allowed on the total. The following problem will illustrate the use of exact interest: (The rate used here is 2%)

		Deposits	Withdrawals	Balances
March	1	600	...	600
	2	...	...	600
	3	200	300	500
	4	100	...	600
	5	300	200	700
	6	300	150	850

	Deposits	Withdrawals	Balances
7	...	200	650
8	...	...	650
9	...	...	650
10	200	...	850
11	...	300	550
12	...	...	550
13	600	450	700
14	300	300	700
15	500	350	850
16	...	...	850
17	400	300	950
18	300	150	1100
19	200	550	750
20	1200	1350	600
21	300	600	300
22	500	100	700
23	...	...	700
24	400	250	850
25	300	100	1050
26	300	200	1150
27	850	300	1700
28	400	150	1950
29	200	...	2150
30	...	...	2150
31	300	650	1800
			28750

One day's interest on $28,750 is $1.5753, or $1.58.

If there are not many items in the month, the use of what may be termed "day-numbers" will shorten the work. In this case a "day-number" is the number of days during which a balance remains unchanged. For instance, if the balance on the seventh of the month is $500.00, and there is no change until the fifteenth, there has been an unchanged balance of $500.00 for eight days, which is obviously equivalent to a balance of eight times $500.00 for one day.

Problem: Find the interest at 2%, exact time, on the following account:

Debits: March 2, $500.00; March 11, $450.00; March 19, $600.00; March 27, $300.00.

Credits: March 8, $300.00; March 14, $200.00; March 28, $500.00.

Date	Debit	Credit	Balance	Day-Numbers	Products
2	500	...	500	6	3000
8	...	300	200	3	600
11	450	...	650	3	1950
14	...	200	450	5	2250
19	600	...	1050	8	8400
27	300	...	1350	1	1350
28	...	500	850	3	2550
Total					20100

One day's interest on $20,100 is $1,1014, or $1.16.

A better method, involving a different use of "day-numbers," will be demonstrated in the next chapter.

PROBLEMS ON CHAPTER II

5. Find the interest on a 3½% Liberty Bond for $50.00 from June 16 to December 15.
6. Find the interest at 6%, bankers' time, on $386.55 from June 17 to October 12.
7. Find the interest at 4½%, bankers' time, on $1,200.00 from July 4 to December 25.
8. A note for $500.00, three months, is dated March 9, interest at 5%. It is discounted May 17 at 6%. Find the net proceeds.

9. Find the interest on daily balances for the month on the following checking account at 2%, exact time.

	Deposits	Checks
1	3000	2000
2	800	500
3	1200	800
4	560	210
5	. . .	. . .
6	380	160
7	650	200
8	1340	620
9	730	400
10	520	260
11	1630	1250
12	. . .	. . .
13	870	1420
14	1130	825
15	760	385
16	920	440
17	1420	675
18	1430	1260
19	. . . .	. . . .
20	1165	1530
21	680	445
22	1480	1260
23	260	420
24	650	725
25	1180	855
26	. . . .	. . . .
27	380	1560
28	640	210
29	360	620
30	1245	1135
31	680	320

10. Find the interest due January 31, at $2\frac{1}{2}\%$, exact time, on these daily balances for the month of January:

Debits: 3rd, \$6,800; 7th, \$4,500; 22nd, \$5,250; 29th, \$1,850.

Credits: 6th, \$4,250; 11th, \$5,300; 20th, \$200; 30th, \$1,200.

CHAPTER III

ACCOUNTS CURRENT

Under the title of this chapter are included two types of problem. The first is the averaging of an account; that is, the statement of the balance as of a certain date, with interest charged and credited at a certain rate. The second is the finding of the equated date; that is, the date on which the balance of the account is at a minimum: when the account can be settled with a minimum payment of interest.

Averaging an account:

Problem: Find the balance of the following account as of July 1, interest at 5% bankers' time:

Debits: April 27, 30-day note without interest, $500.00;
May 18, 60-day note without interest, $600.00;
June 22, cash, $1,200.00.

Credits: June 4, cash, $500.00;
June 18, cash, $200.00.

Of course the 30-day note must be dated from maturity, May 27; and the 60-day note from July 17, which means that when we reckon interest we must deduct 16 days' "discount" for this item.

There are two methods of solution which give the same result. We may reckon interest on each item and balance the account as usual. Or we may simplify the work by using day-numbers.

First solution:

Debit

Amount	Due	Time	Interest
$500.00	May 27	35	2.4304
600.00	July 17	16d	1.3333d
1,200.00	June 22	9	1.50
$2,300.00			2.5971

Credit

Amount	Due	Time	Interest
$500.00	June 4	27	1.875
200.00	June 18	13	.3611
$700.00			2.2361

Balance of principal due..................	$1,600.00
Balance of interest due....................	.3611
Total	$1,600.36

Second solution: In this solution the day-number of each item is the number of days to the balancing date; in other words, the time as in the first solution. In short form the solution is:

500 × 35	17,500	500 × 27	13,500
600 × 16d	9,600 deduct	200 × 13	2,600
1,200 × 9	10,800		
2,300	18,700	700	16,100

Debit balance of day-numbers, 2,600; one day's interest at 5% is .3611, agreeing with the result of the first solution.

When such a statement is made out on the exact interest basis, the advantage of using day-numbers is obvious. Under such conditions the reckoning of a large number of interest items would be a waste of time. By the day-number method we find only one interest item, yet the result is the same to the last decimal place.

Finding the equated date of an account: This has been defined as the date on which the balance of the account is at a minimum. Evidently that is the date on which the interest is at a minimum. Consider any series of transactions. As goods are purchased, interest on the indebtedness begins; as payments are made, the indebtedness is lessened. Between the purchase and the payment there is a date on which the interest on the debt and the discount on the payment equalize each other. A purchase of $500.00 on July 11 and a payment of $500.00 on August 6 would thus equalize each other on July 24, the date half way between. Generally, however, there are many items both debit and credit, and the process of finding this neutral date is complicated. It is necessary to assume some date as a balancing date. This date may be any date whatever, but for convenience it is usual to select either the earliest or the latest date in the problem. This is in order that we may have to perform only one kind of operation; if we use the earliest date, we discount all items to that date; if we use the latest date, we accumulate all items to that date. It seems easier to use the latest date, and the solutions given will all be in that form. After we have balanced the account as of this date, which is called the "focal date," we can find how many days out of the way our focal date is, and reckon backward or forward to the equated date.

An account may be one-sided or two-sided; it may be all debits or all credits, or it may be composed of both debits and credits. We will study a one-sided account first.

Problem: Find the equated date of this account at 6%, bankers' time:

Debits: July 18, $600.00; July 20, $200.00; July 28, 30-day note, $2,800.00.

The latest date is the due date of the note, August 27; we will use that as our focal date.

Amount	Due		Time	Interest
$600.00	July	18	40	$4.00
200.00	July	20	38	1.27
2,800.00	August	27		....
$3,600.00				$5.27

Since both principal and interest are debits, it is evident that the longer the account remains open the larger the balance will be; that is, the equated date must be earlier than the focal date. A balance of $3,600.00 is accruing interest at the rate of 60 cents a day. It has already accrued $5.27. This indicates that interest has been accruing for practically nine days (5.27 ÷ 60). Reckoning backward nine days from August 27, we find the equated date to be August 18. This result can be tested by balancing the account as of August 18.

Another problem will show the solution of a two-sided account, and also the use of day-numbers:

Debits: May 2, 30 days, $1,200.00, interest at 5%.
May 21, $840.00.
June 1, 1,200.00.
Credits: May 15, $600.00.
May 20, $1,900.00.
Interest at 6%, common time.
Focal date, June 1.

Note that the 30-day note is due June 1; but tnat the interest on it cannot appear on the iedger account, and consequently that the amount appearing in the statement is the face of the note, and its date is May 2.
Assume June 1 as focal date.

Due	Amount	Days	Products
May 2	$1,200.00	30	36,000
May 21	840.00	11	9,240
June 1	1,200.00	. .	. . . .
Total	$3,240.00		45,240

Paid	Amount	Days	Products
May 15	$600.00	17	10,200
May 20	1,900.00	12	22,800
Total	$2,500.00		33,000

Balance of day-numbers is 12,240; one day's interest $2.04. Balance of principal is $740.00; one day's interest $12\frac{1}{3}$ cents. Quotient 16.7 or 17 days. Again we have both balances on the same side of the account, so we must reckon backward from the focal date. Seventeen days from June 1 is May 15, the equated date.

Rule for reckoning from the focal date to the equated date:

If balances of both principal and interest are on the same side of the account, reckon backward. This is because, as was explained, the interest on the principal is continually increasing the interest already accrued. But if one balance is debit and the other balance is credit, reckon forward. This is because the interest on a debit principal will tend to extinguish a credit balance of interest, and vice versa.

PROBLEMS ON CHAPTER III

Find the cash balance of each of these accounts:

11: Jan. 1, 6%, bankers' time:

Debits: Oct. 11, $350.00, 30 days; Oct. 22, 30 days, $662.50; Nov. 19, $2,258.00; Dec. 11, 30 days, $350.00.

Credits: Nov. 18, 30-day note without interest, \$600.00; Nov. 22, cash, \$2,000.00

12: July 1, 5½%, exact time:

Debits: May 11, \$252.87 at 30 days; May 13, \$350.00 at 30 days; May 18, \$250.00 at 20 days; June 11, \$353.83 at 10 days.

Credits: May 15, 30-day note for \$500.00, interest at 5%; June 1, cash, \$200.00; June 16, cash, \$200.00; June 28, cash, \$200.00.

13: May 1, 5%, exact time:

Debits: Feb. 27, \$300.00; March 15, \$655.35, 30 days; March 19, \$225.75, 30 days; March 24, \$225.75; April 3, \$625.38, 30 days.

Credits: March 1, 60-day note, with interest at 5%, \$300.00; April 1, 30-day note, without interest, \$1,000.00.

14: Jan. 1, 6%, exact time:

Debits: Oct. 5, 60 days, \$500.00; Oct. 13, 30 days, \$500.00; Oct. 18, 30 days, \$350.00; Nov. 8, 30 days, \$425.00.

Credits: Nov. 18, \$750.00; Dec. 8, 30-day note, with interest at 6½%, \$500.00; Dec. 29, \$200.00.

Find the equated date of each of these accounts:

15: 6%, bankers' time:

Debits: May 17, \$350.00; May 29, \$255.38; June 19, \$125.66; July 5, \$264.87.

16: 6%, exact time:

Credits: Aug. 15, \$325.00; Aug. 19, \$2,703.59; Sept. 14, \$665.32; Oct. 1, \$225.75.

17: 5%, exact time:

Debits: June 25, 30 days, \$1,600.00; July 2, 45 days, \$800.00; Aug. 14, cash, \$500.00.

Credits: July 1, 60 days, \$1,200.00; July 5, cash, \$1,000.00.

18: 5½%, exact time:

Debits: Aug. 14, 30 days, \$300.00; Aug. 22, 45 days, \$500.00; Sept. 17, 30 days, \$225.83; Sept. 29, 30 days, \$246.73.

Credits: Aug. 20, 30 days, \$600.00; Sept. 19, cash, \$250.00.

CHAPTER IV

FOREIGN EXCHANGE

The treatment of foreign exchange in this chapter is wholly from the standpoint of arithmetic. The financial and economic phases of the subject are fully treated in many books on the subject as well as incidentally in most books on banking and foreign trade.

When a sum of money is to be sent from one country to another, it is necessary to change the value from the currency of the sender's country to that of the recipient. This process is the arithmetic of foreign exchange, and the ratio used in making the change is called the rate of exchange.

In all the great commercial nations the monetary system for international trade is based on a gold standard. So theoretically the rate of exchange between any two countries ought to be the ratio between the amounts of gold in their standard monetary units. This theoretical ratio is called par of exchange. For instance, a United States dollar weighs 25.8 grains, and is .9 fine; it therefore contains 23.22 grains of pure gold. The English pound sterling is 113.0016 grains of pure gold. Par of exchange between the two countries is therefore the quotient of these numbers, which is 4.8665. This par of exchange practically never is the actual rate. For the last few years the rate has fluctuated very rapidly and violently, depending on the ravages of the submarines, the varying fortunes of the battlefields all over the earth, the purchases made in this country by England, and many other circumstances.

The following table gives par of exchange between the United States and the other more important commercial nations:

Country	Monetary Unit	Value in U. S. Money
Great Britain	pound sterling	$4.8665
Germany	mark	.2380
France	franc	.1930
Belgium	franc	.1930
Switzerland	franc	.1930
Italy	lira	.1930
Greece	drachma	.1930
Spain	peseta	.1930
Russia	ruble	.5150
Japan	yen	.4980
Holland	guilder	.4000

This theoretical par of exchange is almost never the actual rate, as was said on the previous page. The actual rate is dependent on economic conditions, such as the balance of trade, the supply of gold and its location, labor or other disturbances, and so on. As these vary from day to day, and from hour to hour, so the exchange rate varies.

Rates of exchange are also dependent on the kind of paper to be bought or sold. That is, on the time between the transaction and the day when the cash can actually be collected. The usual kinds of paper are cables, demand, and time: 30, 60, 90 or 120-day paper. Naturally, the longer the time of the paper, the lower the rate; just as in any discount transaction. In the case of a time draft on London, there are also three days of grace to be considered; and the discount or interest is reckoned on these days.

In addition to the actual time of the paper, bankers figure "transit." For instance, a 60-days' sight draft on London made on September 15, may not be presented in London until early in October. It may be held in this country several days waiting for a steamer; the voyage to England occupies several more days; and there may be some delay in securing acceptance. During all this time money is tied up in the paper, and the banker who sold it is entitled to his interest. In this chapter, however, transit will be ignored.

England has always been the leader in the world's trade, and has controlled the gold market. So London has become the center of the world of exchange, and sterling paper has come to be an acceptable medium of exchange in most civilized countries. This means that the great bulk of international settlements are effected in this way. So we will study London exchange first, and the application of the principles demonstrated to other monetary systems will be obvious.

The rate of exchange on London is stated in this country as the dollars and cents value of the pound sterling. It fluctuates by .0005 or fractions or multiples thereof. The smaller English coins are the shilling, 1/20 or .05 of a pound; and the penny, 1/12 of a shilling, which is usually regarded as two cents. The penny is ordinarily indicated by the letter d. Thus, £35 7s. 4d. means 35 pounds, 7 shillings, 4 pence.

Problem: Change £43 11s. 7d. to U.S. money, exchange at $4.855:

£43 11s. = £43.55

43.55 × 4.855	=	$211.44
7 × .02	=	.14
		$211.58

Problem: Change \$368.74 to English money, exchange at \$4.878:

Dividing 368.74 by 4.878 gives		£75.592
$.592 \times 20$	=	11.84s.
$.84 \times 12$	=	10d.

Answer: £75 11s. 10d.

The value of a bill on London depends more or less directly on:

(a) The current rate of demand exchange;

(b) The cost of revenue stamps to be affixed in London;

(c) Interest on the money tied up in the bill;

(d) Commission of the London correspondent.

Problem: What can a banker afford to pay for a bill on London for £355 16s. 3d., at 60 days' sight, demand rate 4.87½, interest 4%, stamp 1/20%, commission 1/4%? One method of solution is to reckon the price per unit of currency, in this case per pound sterling:

Quotation		4.875
From which must be deducted		
Interest for 63 days, exact time	.03366	
Stamp	.00244	
Commission	.01219	.04829
He can afford to buy the bill at		4.82671
times 355.8 and adding .06		\$1717.3434

Exporting gold is sometimes necessary; the gold may be either bars or coin. Such an occasion occurs whenever the exchange market is so deranged that rates are excessively high; that is, whenever the balance of trade is seriously disturbed.

Problem: Find the market quotation on demand exchange at which it is advisable to export gold bars to London if London pays 77s. 10¾d. per ounce for gold 11/12 fine (the English standard). The U. S. Treasury values gold .995 fine at \$20.67183 per ounce.

First change 77s. 10¾d. to pence..... =		934¾d.
This is at 11/12 fine; now convert to the U. S. standard, which is practically 100%, by multiplying by 12/11.......................... =		1019.72727d.
And 1019.72727 ÷ 240.............. = the sterling value of one ounce of pure gold.		£4.248863
Then 20.67183 ÷ 4.248863........... = the dollar value of a pound sterling under those conditions.		4.865263
To this add charges:		
Freight, about 1/8%.........	.006082	
Insurance, 1/20%............	.0024326	
Other charges, 1/20%........	.0024326	
Interest, 6%, 20 days........	.016217	
U. S. bar charge, 40 cents per $1,000.....................	.001946	.0291102
Cost per £ sterling............		4.8943732

That is, if demand exchange is selling at more than 4.894⅜ it will probably be cheaper to pay London in gold bars rather than in sterling drafts.

If gold coin is exported the problem is much the same except that there is no bar charge, but an allowance of about 1/10% must be made for abrasion. We must also keep in mind that the United States gold dollar is .900 fine.

The rate of exchange on **France, Belgium, Switzerland,** and **Italy** is quoted by giving the exchange value of one dollar in francs or lire. Thus, when exchange on Paris is quoted at 5.18, a dollar will buy 5.18 francs, or 5 francs 18 centimes. Exchange on Paris fluctuates by 5/8 of a centime, about 1/8 of a cent. These rates are sometimes further modified by adding or subtracting 1/32% or even

1/64%. Note that the greater the numerical value of the quotation, the lower the value of French currency. A market change from 5.20 to 5.19⅜ is a rise; from 5.20 to 5.20⅝ is a fall.

The processes for changing from French to American currency and vice versa are the exact opposites of those demonstrated for English exchange.

Problem: What is the cost of a bill on Paris for f1,500 exchange at 5.97¾?

1,500 divided by 5.9775 . . . . = $250.94

Problem: What number of francs can be purchased for $250.00 at the same quotation?

250 multiplied by 5.9775 . . . . = f1,494.375

The rate of exchange on **Germany** is quoted by giving the value in cents of four marks. For instance, a quotation on Berlin of .96½ means that each mark is worth 24⅛ cents. The following problems will illustrate the processes:

Problem: How many marks can be purchased for $500.00, exchange at 73¼?

73¼ divided by 4 . . . . = .183125

500 divided by .183125 . . . . = M2,730.38

Problem: What is the proceeds of a German bill for M1,250, at the same quotation?

1,250 × .183125 . . . . = $228.91

South American exchange has always been uncertain, and now that it is coming to be based more on the United States gold dollar no separate treatment is necessary.

Arbitrage of Exchange is the process of making remittances to a country by way of a third country where rates are more favorable. Exchange rates are now so uniform that arbitrage is useful only rarely.

Suppose a banker wishes to remit f25,250 to Paris, exchange at 5.17½. The cost of the draft is then $4,879.23. But if sterling drafts cost 4.84, and London quotes francs at 25.25 per £ sterling, the cost via London is $4,840.00. This is a saving of $39.23.

PROBLEMS ON CHAPTER IV

19. A Boston importer buys goods from a Dresden manufacturer to the value of M21,320. Find the cost of a bill of exchange at 95¾.
20. A bill on London for £342 12s. 6d. is offered at 4.86½. What is the value in dollars?
21. A bill on Paris for f33,250 cost $6,412.72. What was the rate of exchange?
22. A Liverpool merchant draws on an American importer for £540 10s. 6d. What is the value, exchange at 4.85¼?
23. What are the proceeds of a documentary bill of exchange on London for £528 8s. 6d. at 60 days' sight, demand exchange at 4.77½, interest 4%, stamp and commission as usual. Allow 15 days for transit.
24. Tiffany and Co. import an invoice of statuettes from Florence amounting to 14,725.35 lire. They buy a 3-days' sight draft on a Florence banker at 7.35¼, discount 3%, stamp 1/20%, commission 1/8%. Find the cost of the draft.
25. J. P. Morgan and Co. remit to London via Paris when the rates are: New York on London, 4.8675; New York on Paris, 5.19⅜; Paris on London, 25.04. What is the gain on a remittance of £50,000.00?
26. An importer can purchase certain goods in Amsterdam for 15,000 guilders, exchange at 38¾. He can purchase the same goods in Paris for f35,000, exchange at 5.85. Which is cheaper, and how much in United States currency?

27. Find the cost of exporting enough gold to settle a debt of £20,000 if gold bars are shipped to London at 77s. 9½d. per ounce. Charges as on preceding page.

28. A banking concern dealing in foreign exchange has the following transactions on its account with its London correspondent:

Debits:

Sept.	1,	Remittance, 30-day bill,	£400	@ 4.86
	10,	Remittance, sight bill,	£200 10s.	@ 4.87
	15,	Remittance, demand bill,	£200 6d.	@ 4.8675

Credits:

Sept.	2,	Draft, sight, ...	£300	@ 4.87½
	12,	Draft, demand, ...	£200 12s. 5d.	@ 4.87
	20,	Cable, demand, ...	£100	@ 4.88

Ascertain the profit or loss on the account for the month of September, and state the balance as of September 30, in both dollars and sterling, the current rate on that date being 4.89. (Mass. C. P. A., 1914.)

CHAPTER V

POWERS AND ROOTS: LOGARITHMS

If a number is to be multiplied by itself a certain number of times, it is possible to indicate the operation in very brief form. For instance, if it is necessary to multiply 5 together four times, the operation may be indicated thus: 5^4. This expression is then read, "5 to the fourth power." $5 \times 5 \times 5 \times 5$ is 625; and 625 is called the fourth power of 5. In general, 5 is called the base, and 4 is called the exponent, showing how many times 5 is to be multiplied together.

Certain operations of compound interest work can be very much shortened by the use of four elementary laws of exponents. Consider the powers of 10.

$$
\begin{array}{lcr}
10 \times 1 & = 10^1 = & 10 \\
10 \times 10 & = 10^2 = & 100 \\
10 \times 10 \times 10 & = 10^3 = & 1{,}000 \\
10 \times 10 \times 10 \times 10 & = 10^4 = & 10{,}000 \\
10 \times 10 \times 10 \times 10 \times 10 & = 10^5 = & 100{,}000 \\
10 \times 10 \times 10 \times 10 \times 10 \times 10 & = 10^6 = & 1{,}000{,}000
\end{array}
$$

We can now demonstrate the four laws of exponents.

Multiplication: $10^2 \times 10^3 = 100 \times 1{,}000 = 100{,}000$, which is 10^5.

I. Rule: To multiply powers of the same base, add the exponents and keep the base unchanged.

Division: $10^6 \div 10^2 = 1{,}000{,}000 \div 100 = 10{,}000$, which is 10^4.

II. Rule: To find the quotient of powers of the same base, subtract the exponent of the divisor from the exponent of the dividend and leave the base unchanged.

Raising to a power: $(10^2)^3 = 100 \times 100 \times 100 = 1{,}000{,}000$, which is 10^6. Evidently the exponent of the result could be obtained by multiplying 2×3.

III. Rule: To raise a base having an exponent to a given power, multiply the exponents and leave the base unchanged.

Finding a root: The reverse of raising to a power is called finding a root; instead of asking what is the result if we multiply 100 together three times, we ask what was the number which was multiplied together three times so that the result was 1,000,000—in other words, what is the third root of 1,000,000. This question is indicated by the sign $\sqrt[3]{1{,}000{,}000}$. The sign is called a radical sign, and the 3 is called an index. The answer to our question is evidently 100, or 10^2. Since 1,000,000 is 10^6, the result could have been obtained by dividing 6 by 3.

IV. Rule: To find a root of a base which has an exponent, divide the exponent by the index and leave the base unchanged.

Exponents may be fractional, negative, or zero.

Fractional exponents: It is evident that in most cases where a root is to be found, the result will have a fractional exponent. If we have to find the 5th root of the 7th power the result will have the exponent 7/5. In general, the numerator of a fractional exponent indicates a power and the denominator indicates a root. Most of the exponents we deal with are fractional, and for convenience the fractions are reduced to decimals.

Negative exponents: It is also evident that in division the exponent of the divisor might be larger than the exponent of the dividend. This would result in a negative exponent. $10^4 \div 10^6$ is a good illustration. $10{,}000 \div 1{,}000{,}000 = 1/100$. Subtracting exponents, we have as the exponent of the result —2. So 10^{-2} must mean the same thing as 1/100. In general, a negative exponent indicates that the base has become the denominator of a fraction whose numerator is 1. This idea is important in computing present worth.

The zero exponent: $10^2 \div 10^2 = 10^0$ because $2—2 = 0$. Since $100 \div 100 = 1$ it is evident that when the exponent is 0 the base disappears and the value is 1.

Logarithms are an ingenious device for using the rules of exponents. If all the numbers, both whole, mixed and fractional, can be expressed as powers of one and the same base, evidently these rules can be applied in the use of such exponents. Since our number system is decimal, it is natural to select 10 as the base of this system of logarithms. Only a few of these exponents are whole numbers, because only a few numbers are exact powers of 10. The following are a few:

10^6	=	1,000,000	nl	6
10^5	=	100,000	nl	5
10^4	=	10,000	nl	4
10^3	=	1,000	nl	3
10^2	=	100	nl	2
10^1	=	10	nl	1
10^0	=	1	nl	0
10^{-1}	=	.1	nl	−1
10^{-2}	=	.01	nl	−2

and so on. The sign nl is used to separate a number from its logarithm, and is read "the number whose logarithm is". The opposite sign ln "the logarithm of the number" is used in changing from logarithm to number, or antilogarithm, as it is called.

In the table of powers of 10 note that the logarithm of each of the positive powers is one less than the number of figures at the left of the decimal point, while in the negative powers the numerical value of the logarithm is one more than the number of zeros between the point and the first significant digit (a significant digit is any of the digits from 1 to 9).

So far we have spoken only of integral powers: powers whose exponents, or logarithms, are whole numbers. But between 10 and 100, for example, there are 89 whole numbers and an infinite number of mixed numbers. These must all be expressed as powers of 10, and all these powers must lie between 1, the exponent of 10, and 2, the exponent of 100. Evidently these exponents must all be 1+ a decimal. The logarithm of 84, for example, is composed of two parts; a whole number 1, to indicate that it is more than 10, and a decimal to indicate the part of the interval from 10 to 100. The whole number, which indicates the power of 10 just below, is called the **characteristic**; the decimal is called the **mantissa,** or "modifier". Tables of logarithms never give characteristics, because they can be determined by inspection, keeping in mind the table of powers of 10.

Rule for determining the characteristic:

If the number has digits to the left of the decimal point, the characteristic is 1 less than the number of such digits.

If the number is a decimal fraction, with no digits to the left of the point, the characteristic is negative, and is 1 more than the number of zeros between the point and the first significant digit.

For instance, the characteristic of 317.8895432 is 2. The characteristic of .000895 is —4. The characteristic of 7.65 is 0. This last perplexes some students, and it is important that it be thoroughly understood; it is a direct application of the first part of the rule.

Logarithmic tables, then, are wholly tables of mantissae, each one corresponding to a certain series of digits. There are no decimal points, because the characteristic enables us to locate them. It follows that any series of digits always has the same mantissa. In the tables, for instance, the mantissa corresponding to 8427 is 925673. That is

8427	nl	3.925673
842.7	nl	2.925673
84.27	nl	1.925673
8.427	nl	0.925673
.8427	nl	$\bar{1}.925673$

and so on. This last logarithm is also written 1.925673, and 9.925673—10. This last method will be followed in this book, as it simplifies certain processes which are quite common in annuity work. The logarithm of .000895 on the preceding page would be written 6.951823—10.

As was stated in the first chapter, logarithms should run to at least six places, and seven are better. The first two digits of the mantissa change slowly, and are indicated only when they change. This is an important fact to remember in using tables, because the first two digits are apt to be several lines above the others, and must be found and used in order to get correct results.

Finding a logarithm in the tables is a simple matter. The digits of the antilogarithm are in a column at the left of the page and across the top. To find the logarithm or rather the mantissa of 357.4 turn to the page or pages containing the 3,000 series, and follow down the left margin until 357 is reached. Across the top are the digits from 0 to 9, and in the 4 column, opposite 357 we read 3,155. Remembering that the first two digits must be taken also, we read just above 55. The mantissa is then 553,155. Now determine the characteristic, which in this case is 2, and the entire logarithm is 2.553155. If there are less than four digits in the antilogarithm, affix zeros. For the mantissa of 37 look for 3,700; the logarithm is 1.568202.

Usually the problem is to find the mantissa of a number having more than four digits. This is known as **interpolation.** To find the logarithm of 231.24. The characteristic is 2. The mantissa is between the mantissae of 2,312 and 2,313, and evidently is 4/10 of the interval.

2,313	nl	364,176
2,312	nl	363,988
difference		188

4/10 of 188 is 75.2, which, added to 363,988, gives 3,640,632. Prefixing the characteristic we have 231.24 nl 2.3640632.

The cologarithm of a number is the logarithm of its reciprocal. We met a reciprocal is discussing negative exponents, and remarked in passing that the commonest occurrence is in finding present worths. A reciprocal is 1 divided by a number. Now the logarithm of 1 is 0, as was explained on page 21. To find the cologarithm of 387.2, or the logarithm of 1 divided by 387.2:

1	nl	10.000000—10
387.2	nl	2.587935
Subtracting		7.412065—10

One other very common use for cologarithms is in examples where multiplication and division must be performed together. The division can be performed by adding the cologarithm of the divisor, as will be demonstrated presently.

Finding the antilogarithm is somewhat more difficult, because it almost always requires interpolation. To find the number whose logarithm is 2.7851. The characteristic 2 indicates that there are three digits at the left of the decimal point. Looking in the tables for mantissae nearest 7,851 we find:

785,116	ln	6,097	
7851			the given mantissa
785,045	ln	6,096	

The difference between the logarithms in the table is 71; the given logarithm is 55 more than 785,045. The required antilogarithm is therefore 55/71 of the interval from 6,096 to 6,097. 55/71 of 1 is .774; affixing this to 6,096 we have 6,096,774, and placing the decimal point after the third digit, we have the antilogarithm 609.6774. The following problems will illustrate all the logarithmic processes which are necessary for this book.

Multiplication: 387.2 × .04752:

	387.2	nl	2.587935
	.04752	nl	8.676876−10
Adding			11.264811−10
or			1.264811
ln			18.3997

It is very commonly necessary to add 10−10 or multiples of it, or to subtract them. Since 10−10 is 0 we evidently do not change the value of any expression by this device. The addition of the logarithms is in accordance with the first law of exponents, in multiplication, page 33.

Division: 69.45 ÷ 3.894

69.45	nl	1.841672
3.894	nl	.590396
Subtracting		1.251276
ln		17.835123

The subtraction is in accordance with the second law of exponents, in division.

Multiplication and division in the same example:

$$\frac{68.39 \times 31{,}750}{243.6}$$

68.39	nl	1.834993
31750	nl	4.501744
243.6	n—colog	7.613323—10
Adding		13.950060—10
or		3.950060
ln		8913.7347

Raising to a power:

a) $(2.798)^3$

2.798	nl	.446848
times 8		3.574784
ln		3756.504

b) to illustrate treatment of a negative characteristic $(.0693)^5$

.0693	nl	8.840733—10
times 5		44.203665—50
or		4.203665—10
ln		.0000015983

Multiply by the exponent; see rule III.

Finding a root:

a) $\sqrt[7]{968.4}$

968.4	nl	2.986055
divided by 7		.4265793
ln		2.67042

b) to illustrate treatment of a negative characteristic $\sqrt[4]{.08355}$

.08355	nl	8.921946—10
after we divide by 4 the characteristic must be —10; so before division it must be —40; so it is necessary to add 30—30, making the log		38.921946—40
now divide by 4		9.7304865-10
ln		.537634

Divide by the index as in rule IV, page 34.

Combinations of these processes are rarely necessary.

a) $\dfrac{(3.066)^{11}}{\sqrt[5]{12.74}}$

3.066	nl	.486572	
times 11			5.352292
12.74	nl	1.105169	
divided by 5			.2210338
Subtracting			5.1312582
ln		153287.6	

b) $\dfrac{4.925 \times \sqrt[3]{16740}}{(38.25)^4}$

4.924	nl		.692406
16740	nl	4.223755	
divided by 3			1.407918
38.25	nl	1.582631	
times 4		6.339524	
divide by adding colog			3.669476—10
Adding			5.769800—10
ln			.0000588573

PROBLEMS ON CHAPTER V

A. Problems to give facility in use of tables:

29. Find the logs of 432.5 4.325 .004325
30. Find the logs of 5700 5.7 570000
31. Find the logs of 78.9342 95.4386 7762.47
32. Find the cologs of 13.97 .04953 620000
33. Find the cologs of 6.91843 100.764 .000015
34. Find the antilogs of 3.270679 1.284656 8.817962—10
35. Find the antilogs of 1.316274 2.967777 4.830034—10
36. Find the antilogs of .00317255 6.779999—10 3.84506234

B. Perform the following by the use of logs:

37. Multiplication:

378.2×56.43

$.06925 \times 34700$

$616500 \times 3.9483 \times .00745$

38. Division:

$930.07 \div 1.05$

$.58023 \div 7.7438$

$58.023 \div .077438$

$8.9094 \div .89094$

39. Multiplication and division:

$$\frac{9.1932 \times .5307}{.0076}$$

$$\frac{6800000 \times .0038902}{1.025}$$

40. Raising to a power:

$(3.4684)^6$

$(1.01225)^{14}$

41. Raising to a power:

$(.6307)^9$

$(.1213)^6$

42. Finding a root:

$$\sqrt[3]{810.5}$$

$$\sqrt[8]{1.0575}$$

43. Finding a root:

$$\sqrt[9]{.0069183}$$

$$\sqrt[5]{.0003301}$$

Miscellaneous:

44. $$\frac{4.6173 \times (2.837)^4}{.10765 \times 48.27}$$

45. $$\sqrt[5]{\frac{3 \times 7.5}{(1.015)^4}}$$

46. $$\frac{\sqrt[3]{392.7} \times 61.3455 \times 33.9}{(1.0175)^6}$$

47. $$\left[\frac{2.94635 \times \sqrt[3]{67}}{1.023}\right]^6$$

48. $$\frac{\sqrt[4]{300.75 \times .000045}}{(.8512)^8}$$

CHAPTER VI

COMPOUND INTEREST AND PRESENT WORTH

At this point we enter on the study of what the American Institute of Accountants calls Elementary Actuarial Science. An actuary is the mathematical expert of a life insurance company. It is his business to calculate the income and expenditures of the company and see that the company is on a safe financial basis: to see that it has sufficient funds on hand to meet all ordinary calls and an additional amount to meet emergencies. These expenditures are concerned mostly with payments on account of policies, either life or endowment. Such expenditures therefore can be estimated pretty closely if there are sufficient data regarding the number of persons who will live or die during the year. In other words, the actuary is concerned with the probabilities, as they are called. By the study of many thousand lives actuaries have been enabled to formulate laws stating that out of a certain number of persons alive on a given date a stated proportion will live through an entire year and the rest will die within the year. These contingencies, so called, affecting sometimes a single life and sometimes a group of lives, make the mathematics of the actuary exceedingly difficult.

Moreover, these calculations must be based on a long period of years. The mortality tables for straight life policies run to the ninety-sixth year. Most endowment insurance is for periods of ten to twenty-five years. Under these circumstances all calculations must be on a compound interest basis. The series of payments on an endowment insurance policy are invested by the company and earn interest. It is reasonable that the income on the payments of each policyholder should be credited to his policy.

The mathematics which have been outlined in the last two paragraphs are Actuarial Science. The effect of probability on annuities means nothing to the accountant, however, because he assumes that the financial arrangements which he supervises will be permanent in most cases. To describe this situation, mathematicians speak of annuities certain; annuities which will not be affected by the death or mishap of any person or persons, but will continue to the limit of the time assigned them. This is Elementary Actuarial Science.

A very common way of referring to compound interest is "interest on interest". This definition is not true, and is misleading. It is misleading because it implies usury. No such taint inheres in compound interest; every dollar of capital is always at work, and inasmuch as interest rates are stated for definite periods, a year, six months, three months, at the end of the period the interest which has been accruing becomes due and is no longer interest, but an addition to capital. So again it is not interest on interest, but interest on a periodically increasing capital. The chief argument for compound interest is this. If a sum of money is invested, the interest is supposed to be paid when due. It is loaned for the sake of this income. If the interest is not paid promptly, part of the income is lost and the interest rate is lowered. But this interest is earning money for the party with whom the money is invested. So he and not the rightful owner is reaping the benefit of the investment.

What is the significance for the accountant? In the first place, many of the transactions of the business world are already on a compound interest basis. This applies especially to investment accounting; to bond accounts and businesses, savings banks, insurance companies, and even the interest on checking accounts in commercial banks. And many accountants in supervising bond issues of public and private corporations say that they build the sinking fund schedules on a compound interest basis whenever the accounting staff of the concern are capable of handling schedules built in that way.

Evidently, if an accountant is to audit the accounts of such a concern he must know his annuities thoroughly.

Again, as has been said already, every business has its fiscal period, and at the end of that period, the net earnings are transferred to capital in any one of a number of ways. If the investment in the business is already adequate, any overplus of capital will be invested in other businesses or in securities, or at least in some banking house which offers a fair rate of interest. In some way this money will earn an income at not less than the savings bank rate.

Usually an additional investment in the business is desirable, which may take the form of a direct increase in surplus, or may be apportioned among the various reserves, for sinking funds, depreciation, or so on. Such additional investments of course earn an additional income, just as the previous investment did. If the business man expected an 8% return on his original investment he will hope for an 8% return on the increase. And it seems reasonable that such income should be credited to the surplus or reserve on which it was earned. Because if such reserves were represented by funds in the hands of a trustee, they would earn an income at not less than the savings bank rate, and this income would not be transferred to the free surplus of the company, but would be added to the fund in the hands of the trustee and reinvested by him.

These are the arguments for compound interest. They are the arguments from reason and consistency and from the best custom. Failure to recognize them causes misstatement of income and failure to allocate it properly between the various parts of free or appropriated capital on which it was earned. Compound interest is actually at work whether realized or unrealized; acknowledgment of it as a fact is the duty of the progressive accountant.

Computation of compound amount: We have been discussing compound interest. As a matter of fact we usually find the compound amount first, and then deduct the principal, the result being the interest. The reason for this is that there is a very simple process for finding the compound amount of $1.00 for any time at any rate; and since all dollars in a given transaction increase at the same rate, the compound amount of any principal is merely the compound amount of $1.00 multiplied by the given principal.

We have perhaps been accustomed to find compound amount by a laborious process, thus:

To find the compound amount of $1.00 at 5% for five years:

1st year	$1.00
add 5%	.05
2nd year, principal	1.05
add 5%	.0525
3rd year, principal	1.1025
add 5%	.055125
4th year, principal	1.157625
add 5%	.05788125
5th year, principal	1.21550625
add 5%	.06077531
Compound amount	$1.27628156

Of course no savings bank or savings department of a commercial bank carries its interest to eight places of decimals. But this must be done in investment mathematics, where the items which are being accumulated may run into millions of dollars.

A much simpler way to solve the problem is given by an elementary principle of algebra. The principal is 1; the rate of interest, hereafter called i, is .05. The amount at the end of one period is, therefore, 1.05. We are compounding annually, so the number of conversion intervals, as they are called (the number of times that the interest becomes an addition to principal) is 5; this time element is indicated by n. Now if we multiply 1.05 together 5 times, or, as the mathematicians say, raise it to the fifth power, the product will be $1.27628156, the very same result as we obtained by the other method. In general, to find the compound amount of 1 for n periods at rate i, raise 1 + i to the n-th power. Expressed as a formula this is $(1 + i)^n$. This is indicated commonly by the symbol s^n.

It is important to remember that n is the number of conversion intervals, not the number of years. For instance, if a savings bank states that its rate is 4%, it usually means 2% every six months. A dollar left in this bank five years will amount to the tenth power of 1.02.

For ordinary times and rates there are good tables of compound amount.

Problem: To find the compound amount of $150.00 for 17 years at 5% compounding semiannually. The rate is 2½% and the time is 34 periods. In the 2½% column of the tables opposite 34 of the time column read, 2.31532213. Multiplying by 150 gives the compound amount $347.2983. The compound interest is $197.2983.

For unusual rates, which often occur, logarithms are necessary. And they should be at least ten-place logarithms, as was stated in the first chapter.

Problem: Find the compound amount of \$246.75 for 12 years at 2⅜% compounding annually.

i is .02375		
1 + i is 1.02375		
n is 12.		
We have to find $(1.02375)^{12}$		
1.02375	nl	.0101939148
times 12		.1223269776
ln		1.325338
times 246.75 = \$327.027		

Nominal and effective rates: At the beginning of the chapter on interest one element of the interest problem mentioned was "frequency", the length of the "conversion interval". For instance, savings banks and departments state a yearly rate, but compound semiannually or even monthly. Bonds usually pay semiannual interest, and this is considered when the price is quoted. These are only two common instances out of many. When a bank advertises compound interest at 4%, convertible semiannually, it usually means to pay 2% every six months; the actual yearly rate is 4.04%. In this case 4% is called the nominal rate, indicated by j, and 4.04% is called the effective rate, indicated by i. If the compound amount of \$1.00 for five years at 4% nominal, convertible semiannually, is desired, i is .02, 1 + i is 1.02, and n is 10. In the table of s^n in the 2% column opposite 10 read 1.21899442. Similarly, the compound amount of \$1.00 for 15 years at 6% nominal, convertible quarterly, is the compound amount for 60 periods at 1½%, or 2.44321978. Such problems usually lead to unusual rates and require the use of logarithms in solution.

Evidently, given a nominal rate, the shorter the conversion interval, the higher the effective rate.

At 6% compounding	yearly	$(1 + i)^n$ is	1.06
"	semi-annually		1.0609
"	quarterly		1.061364
"	monthly		1.061678
"	daily		1.061826

There is a limit to this increase, however. For the nominal rate 6%, the highest possible effective rate, for instantaneous compounding, is slightly less than 6.184%. This idea of instantaneous compounding, or continuous compounding, is treated at length by the books on Actuarial Science. The rate 6.184% is called "force of interest". It has no significance for the accountant.

To change from nominal to effective rates:

The formula is $i = \left[1 + \frac{j}{m}\right]^m - 1$

where i is the effective rate, j is the nominal rate, and m is the number of conversion intervals in one year.

Rule: Find the compound amount of the periodic rate for m periods and subtract 1.

Problem: What is the effective rate when the yearly nominal rate is 3%, and compoundings are semiannual?

j is .03

m is 2

$\frac{j}{m}$ is .015

$1 + \frac{j}{m}$ is 1.015

Then the effective rate is $(1.015)^2 - 1$, or .030225. Stated as a percent it is 3.0225%.

Nominal rate when the effective rate is given: Insurance companies and some other concerns which have a great variety of investments having varying interest dates and periods find it advisable to annualize all interest rates. This leads to a situation which is rather difficult to grasp at first. They assume a yearly rate of income, say 6%. Now if a certain investment pays semiannual interest and is to be valued at 6% annually, it is evident that 6% is the effective rate. What is the semiannual rate? Not 3%, for that would yield an annual rate of 6.09%, as was stated on the previous page. Referring to the discussion of roots on page 20, the rate which after two conversions yields 1.06 must be the square root (or second root) of 1.06. If $(1 + i)^2 = 1.06$, then $1 + i = \sqrt[2]{1.06}$. This solution of such a problem is logarithmic and based on the fourth rule of exponents.

1.06	nl	.0253058653
dividing by 2		.01265293265
ln (10-place table)		1.029563
or		2.9563% per period
that is		5.9126% is the nominal rate.

Such a nominal rate is indicated by the symbol j_p, where p indicates the number of compoundings per year. A table of the values of j_p for the usual rates and for two, four and twelve compoundings yearly is given in all good collections of tables.

To illustrate a little differently, the value of j_4 for 5% is given in that table as .0490889. This is a nominal rate, which means that the periodic rate is 1/4 of the annual rate. Now 1/4 of .0490889 is .0122722. Then, according to the definition of a periodic rate given, 1.0122722 raised to the fourth power ought to yield 1.05. By actual multiplication we find the rate to be 1.0500025. The error is of course due to rounding up whenever digits were discarded.

Formula for finding the nominal rate when the effective rate is given is

$$j_p = p\left(\sqrt[p]{1+i} - 1\right)$$

Rule: Find the p-th root of $1 + i$ and subtract 1; this is the periodic interest rate; multiply by p to find the annual nominal rate.

To sum up, rates are always given on a yearly basis. If compoundings are other than yearly, there may be two ways of stating the rate. Either the rate given is a nominal rate, in which case the periodic rate is the n-th part of the given rate. Or the rate may be effective, in which case the periodic rate is the n-th root of the given rate.

Present worth and compound discount: If a sum of money is due at some future date, it can be discounted to the present time by deducting interest from the present to the due date. This is a common process on short time notes, in which case the interest is simple interest and the process a very simple one. But in the case of long time transactions, where compound interest must be reckoned, the process is somewhat different. There is an old principle of arithmetic that if the product of two numbers is known and also one of the factors, the other factor can be found by division. Since compound interest is found by multiplication, evidently present worth on a compound interest basis is found by division. And since all our compound amounts are based on 1, and expressed by the formula $(1 + i)^n$, evidently present worth of 1 is 1 divided by $(1 + i)^n$, or expressed as a formula,

$$\textbf{present worth of 1} = \frac{1}{(1+i)^n}$$

Rule: Divide 1 by the compound amount of 1 for the given time and rate.

Present worth is indicated by the symbol v^n. Tables for the usual rates and times are given in all good collections of tables.

Problem: Find the present worth of \$125.00 due in 12 years on a 6% nominal basis, convertible semiannually.

1 is 3%
n is 24
In the table of v^n, 3% column, opposite
24 read .49193374
times 125 \$61.4917.

Problem: (to illustrate use of logarithms when the rate is unusual): Find the present worth of 1 due in two years at 3¼% nominal, convertible semiannually.

i is .01625
n is 4

1.01625	nl	.0070005586
times 4		.0280022344
colog		9.9719977656—10
ln		.937557

Note that since the $(1 + i)^n$ is in the denominator, the value of v^n always requires finding of a cologarithm.

SUMMARY OF THE PROCESSES USED IN COMPOUND INTEREST COMPUTATION

Compound amount: $s^n = (1 + i)^n$.

Problem: Find the compound amount of \$1.00 at 4½% nominal, convertible semiannually, for 20 years.

1 = 2¼%
n = 40

1.0225	nl	.0096633167
times 40		.3865332668
ln		2.435189

Present worth: $v^n = \dfrac{1}{(1 + i)^n}$

Problem: Find the present worth of $1.00 due in 22 years at 3½% nominal, convertible semiannually.

i = 1¾%		
n = 44		
1.0175	nl	.0075344179
times 44		.3315143876
colog		9.6684856124—10
ln		.466107

Present worth of an interest bearing debt:

Given a debt due in n periods with interest at rate k per period. To discount it to the present time at rate i per period.

There are two cases, one when the conversion intervals at rates k and i are the same, and the other case when the conversion intervals differ. Let m be the number of periods in the term of discount.

The formula is $v = \dfrac{(1 + k)^n}{(1 + i)^m}$

Rule: Find the compound amount of 1 at rate k, and divide by the compound amount (or multiply by the present worth) of 1 at rate i.

Problem: A debt of 1 bears interest at 5% annually, and is due in 12 years. What is the present worth at 6% annually.

$(1.05)^{12}$	=	1.79585633
$\dfrac{1}{(1.06)^{12}}$	=	.49696936
product		.8924856

Problem (to illustrate differing conversion intervals): Find the present worth of $155.63 due in 5 years with interest at 5% nominal, convertible semiannually, discounted at 6% annually:

(1.0125)	=	1.28008454
$\frac{1}{(1.06)^5}$	=	.74725817
product		.9565536
times 155.63	=	\$148.86843

To change from nominal rate to effective:

The formula is $i = (1 + \frac{j}{m})^m - 1$

Problem: What is the effective rate when the stated rate is 6%, compounded monthly.

j	=	6%	
m	=	12	
$\frac{j}{m}$	=	.005	
1.005		nl	.0021660618
times 12			.0259927416
ln			1.0616774
subtract 1			.0616774
or			6.16774%

To change from effective to nominal:

The formula is $j_p = p\,(\sqrt[p]{1 + i} - 1)$

Problem: What is the nominal rate equivalent to an effective rate of 4¼%, if there are four conversion intervals in the year.

1.0425	nl	.0180760636
divided by 4		.0045190159
ln (10-place table)		1.0104597
subtract 1		.0104597
or		1.04597% periodic rate
times 4		4.18388% yearly

To find the rate, given principal, amount, and time:

The formula is $i = \sqrt[n]{\frac{S}{P}} - 1$

Rule: Divide the amount by the principal; find n-th root of the quotient; subtract 1.

Note that usually the easiest way to divide S by P is to find the difference of the logarithms.

Problem: At what annual rate will \$1.00 amount to \$2.00 in 27 years.

S divided by P	=	2.
2	nl	.30103
divide by 27		.0111492555
ln (10-place tables)		1.026045
subtract 1 and the rate is		2.6045%

Problem: At what nominal rate, compounding quarterly, will \$3.00 amount to \$5.00 in 10 years.

5	nl	.69897
3	nl	.477121
subtract		.221849
divide by 40		.005546225
ln		1.012853
or		1.2853% periodic rate
times 4		5.1412% annually.

To find the time, given principal, amount, and rate:

The formula is $n = \frac{\log S - \log P}{\log (1 + i)}$

Rule: From log of S subtract log of P; divide by log of 1 + i.

Problem: In what time will \$2.00 amount to \$3.00 at 5% annually?

3	nl	.477121
2	nl	.301030

subtract		.176091
$1 + i = 1.05$	nl	.021189
quotient		8.31 years.

The quotient should not be changed to months and days, because compound interest is not assumed for parts of periods. In this type of problem, and whenever the logarithm of $1 + i$ is used as a divisor, the 6-place mantissa is sufficient; if the 10-place mantissa were used it would be cumbersome and we would begin to cut off digits at once. Note also that although one logarithm is divided by another, the quotient is not a logarithm. This situation always occurs in time-problems.

When the rate is nominal, it is necessary to proceed as follows:

Problem: In what time will $1.00 amount to $2.00 at 5% nominal, convertible quarterly?

periodic rate is		.0125
2	nl	.301030
1	nl	.000000
subtract		.30103
1.0125	nl	.005395
quotient		55.8 periods
or		13.95 years

Compare this with the approximation commonly used,

$$n = \frac{.693}{i} + .35$$

$$\text{in this case } \frac{.693}{.0125} + .35 = 55.79 \text{ periods.}$$

PROBLEMS ON CHAPTER VI

A. Simple problems to be solved by use of the tables and multiplication or division:

49. What is the compound amount of $1.00 for 15 years at 6% annually?
50. What is the compound amount of $1.00 for 15 years at 6% nominal, convertible semiannually?
51. What is the compound amount of $383.97 for 16¼ years at 5% nominal, convertible quarterly?
52. What is the compound amount of $1,500.00 for 22 years at 6% nominal, convertible quarterly.
    (Note: If the table does not include values for 88 periods, they can be obtained by multiplying the value for 85 periods by the value for 3 periods. See the first law of exponents.)
53. What is the compound amount of $2,785.94 for 20 years at 3% nominal, convertible semiannually?
54. What is the effective rate equivalent to 3¾% nominal, convertible quarterly?
55. What is the present worth of $1.00 due in 17 years at 6% annually.
56. What is the present worth of $1.00 due in 17 years at 5% nominal, convertible quarterly?
57. What is the present worth of $1,934.17 due in 11 years at 6% nominal, convertible semiannually?

B. More difficult problems, usually requiring the use of logarithms in solution:

58. What is the compound amount of $1.00 for 11 years at 4½% nominal, convertible quarterly?
59. What is the compound amount of $1.00 for 16 years at 6% nominal, convertible monthly?
60. What is the compound amount of $1.00 for 11 years at 4¼% nominal, convertible semiannually?

61. What is the effective rate equivalent to $3\frac{3}{4}\%$ nominal, convertible monthly?
62. What is the effective rate equivalent to $7\frac{1}{2}\%$ nominal, convertible quarterly?
63. What nominal rate, compounded quarterly, is equivalent to 6% effective?
64. What nominal rate, compounded monthly, is equivalent to $5\frac{1}{4}\%$ effective?
65. What is the present worth of $1.00 due in 11 years, at $4\frac{1}{4}\%$ nominal, convertible semiannually?
66. What is the present worth of $1.00 due in 8 years at 3.9% nominal, convertible monthly?
67. At what annual rate will $15,000 double itself in 25 years?
68. At what nominal rate, convertible quarterly, will $3,000.00 amount to $5,000.00 in 5 years?
69. The British government sold non-interest bearing certificates for 15s. 6d., redeemable after 5 years at £1. What is the equivalent savings bank rate, nominal, convertible semiannually.
70. Benjamin Franklin bequeathed $5,000.00 to the City of Boston. After 104 years it had amounted to $431,000.00. What is the equivalent rate, nominal, convertible quarterly?
71. In what time will $1.00 double itself at 6% annually? Solve by logs and check the solution by the approximation formula.
72. $5,000.00 is invested at 5% nominal, convertible semiannually. In what time will it amount to $7,500.00?

CHAPTER VII

ANNUITIES: AMOUNT AND PRESENT WORTH

An annuity is a series of equal payments at equal intervals of time, accumulated at a fixed rate of interest. The amount of the periodic payment is always stated on a yearly basis. A good example of this is the interest on a bond. The bond is advertised as a 5% bond; but the issuing or selling concern means that the bond will pay 2½% every six months. Another important fact to remember is that payments occur at the end of the year or other period. This is reasonable, as will appear if we consider the source of the money with which the payments are made. A corporation pays the interest on its bonds out of income. This income is reckoned at the end of the fiscal period. A person paying interest on a mortgage does so at the end of the interest period. There are numerous other instances which will occur to any accountant. The most common instances of payments at the beginning of the period are premiums on insurance policies and some rental payments. Such an annuity is called an annuity due and calls for special treatment. There is also the deferred annuity, in which no payments are made until after a specified time. These types will be taken up after the ordinary annuity has been studied.

The amount of an ordinary annuity is indicated by s_{n}. This must always be carefully distinguished from s^{n}, which is the symbol for compound amount.

Problem: Find the amount of an annuity of five annual payments of $1.00 each, accumulated at 4% annually.

First we will establish the value by accumulating the payments singly. Then we will learn a much simpler way of arriving at the same result.

First	payment,	accumulated	4	periods,	$1.16985856
Second	"	"	3	"	1.12486400
Third	"	"	2	"	1.08160000
Fourth	"	"	1	"	1.04000000
Fifth	"	, cash			1.00
Total					$5.41632256

The better way is this: The compound **interest** on $1.00 for 5 years at 4% annually is .2166529. This amount divided by the single interest .04 gives the same result as above, $5.4163225.

Remembering that s^n or $(1+i)^n$ is the symbol for compound amount, it is evident that compound interest is s^n-1, or $(1+i)^n-1$. So we have the

Formula for the amount of an annuity:

$$s_n = \frac{s^n-1}{i} = \frac{(1+i)^n-1}{i}$$

Rule: Divide the compound **interest** by the interest for a single period.

Tables of s_n for the usual rates and times are found in all good collections of tables.

In accounting practise, whenever an annuity occurs, a schedule should be set up at once covering the entire life of the annuity. This schedule serves first as a check on the accuracy of the computation, and second as the source of the periodic Journal entries necessitated by the interest or other elements of the annuity.

Problem: To prepare a schedule of a series of 5 annual payments of $1,000.00 each, improved at 4% annually. We have just seen that the amount of such an annuity of $1.00 is $5.4163226. For $1,000.00 the amount is evidently $5,146.3226, as is proved by the following:

Schedule of Accumulation of Annuity of Five Annual Payments of $1,000.00 at 4% Annually.

Year	Interest 4%	Annual Payment	Total Addition to Fund	Amount in Fund
1		$1,000.00	$1,000.00	$1,000.00
2	$ 40.00	1,000.00	1,040.00	2,040.00
3	81.60	1,000.00	1,081.60	3,121.60
4	124.864	1,000.00	1,124.864	4,246.464
5	169.8586	1,000.00	1,169.8586	5,416.3226
Totals	416.3226	5,000.00	5,416.3226	

Such a schedule should always be footed down as a proof of the correctness of the additions. Total Interest plus total Payments must equal the footing of Addition to Fund, and also the last item in the Amount in Fund column. Also, each item in Total Addition to Fund multiplied by $1 + i$ must give the next item.

If the schedule is lengthy it is a wise precaution to prove every five or at most ten years. The first method of proof is by testing footings as has just been explained. A better method is to find the amount which should be in the fund at the time. The proper value of s_n multiplied by the periodic payment must equal the amount in the fund at the time.

Annuity problems often require logarithmic solution. For this purpose the second formula given is the most easily understood.

Problem: Find the amount of an annuity of $1.00 a year for 20 years at 4¼% annually.

The compound amount must first be found.

1 + i = 1.0425	nl	.0180760636
times 20		.361521272
ln = s^n =		2.2989062
subtract 1		1.2989062 compound interest

divide by single interest .0425
gives $30.5625

Effective rates: So far all our calculations have been based on annual or nominal rates. Sometimes, however, the effective rate is stated, but there are several payments a year, and there are as many conversion intervals as payments. There are two methods of attacking this problem.

The first method is to reckon the value of the periodic payment so that nominal rates can be used. Suppose an annuity of $1.00 at 4% effective, payments to be made quarterly. If we assume that the quarterly rate is 1%, the payment will be slightly less than 25 cents, because four payments of 25 cents improved at 1% would amount to more than the specified $1.00 at 4%. Tables of the value of such periodic payments are sometimes given.

Another method involves the use of j_p. Suppose an annuity of $1,000.00 for three years, payments semiannual, at 3% effective. If we assume that the payment is $500.00, we must accumulate at such a rate as is equivalent to 3% annually. The table of j_p for 3% semiannually gives .0297783; that is, the semiannual rate is 1.4889%. It is worth while to show the proof schedule in full. This schedule will be presented in columnar form:

End of first	period,	payment	$500.00
" second	"	interest	7.4445
		payment	500.00
		Total	$1,007.4445

"	third	"	interest	14.9998
			payment	500.00
			Total	\$1,522.4443
"	fourth	"	interest	22.6676
			payment	500.00
			Total	\$2,045.1120
"	fifth	"	interest	30.4495
			payment	500.00
			Total	\$2,575.5615
"	sixth	"	interest	38.3475
			payment	500.00
Total				\$3,113.9090

Whenever tables can be used, the value found in the tables should be multiplied by the quantity $\frac{i}{j_p}$. The effect of this is to cancel i out of the denominator of the formula for s_n and put j_p in its place. A table of the values of $\frac{i}{j_p}$ is given in all good collections of tables.

In the problem for which the schedule appears on the preceding page, the table value for a 3-year annuity of 1 at 3% annually is 3.0909. The value of $\frac{i}{j_p}$ when i is 3% and $_p$ is 2 is 1.0074446. The product of 3.0909 × 1.0074446 × 1000 is 3113.91, as before.

Whenever logarithms are used, the compound interest should be divided by j_p instead of i. In such cases both compound amount and j_p have to be found by use of logarithms.

Problem: Find the amount of an annuity of \$1.00 for 5 years at 3¼% effective, the annuity being payable semi-annually.

First we will find the compound interest:

1.0325	nl	.0138900603
times 5		.0694503015
ln $=$ s^n		1.1734116
less 1		.1734116

Next we will find j_p:

1.0325	nl	.0138900603
divided by 2		.00694503015
ln		1.01612
less 1		.01612
times 2 $= j_p$		.03224
The quotient is		\$5.3787

Formulae for use of j_p in finding the amount of an annuity:

With tables: s_n times $\frac{i}{j_p}$

With logarithms: $\frac{(1+i)^n - 1}{j_p}$

In both cases **n is the number of years.**

The present worth of an ordinary annuity is indicated by the symbol a_n. We will demonstrate the formula for finding the present worth just as we demonstrated the amount.

Problem: Find the present worth of an annuity of \$1.00 for 5 years at 4% annually.

First	payment,	discounted	1	year	.96153846
Second	"	"	2	"	.92455621
Third	"	"	3	"	.88899636
Fourth	"	"	4	"	.85480419
Fifth	"	"	5	"	.82192711
Total					\$4.45182233

A quicker way is this: The present worth of $1.00 due in 5 years at 4% annually is .82102711; therefore the **compound discount** is .17807289. This, divided by the single interest .04, gives 4.45182225 as above, allowing for inaccuracies in the right-hand digits.

Remembering that v^n is the symbol for present worth, $1-v^n$ is the symbol for compound discount. So we have the formula for present worth of an annuity:

$$a_n = \frac{1-v^n}{i}$$

For logarithmic computation this formula may be better stated

$$a_n = \frac{1-\dfrac{1}{(1+i)^n}}{i}$$

Rule: Divide the compound **discount** for the given rate and time by the single interest.

Tables of a_n for the usual rates and times are found in all good collections of tables.

The schedule in tabular form is necessary whenever such a situation occurs. For the annuity of $1,000.00, 5 years at 4% annually, the value of a_n is $4,451.8225.

Schedule of Amortization of a Debt of $4,451.8225 Payable in Annual Instalments of $1,000.00, with Interest on Outstandings at 4% Annually

Year	Interest 4%	Annual Instalment	Net Amortization	Balance Outstanding $4,451.8225
1	$178.0729	$1,000.00	$821.9271	3,629.8954
2	145.1958	1,000.00	854.8042	2,775.0912
3	111.0036	1,000.00	888.9964	1,886.0948
4	75.4437	1,000.00	924.5563	961.5385
5	38.4615	1,000.00	961.5385	
Totals	$548.1775	$5,000.00	$4,451.8225	

The usual checks should be observed. Total Interest deducted from total Annual Payments leaves total Net Amortization, which is the same as the amount of the debt at the beginning. If the schedule runs to twenty years or more it should be tested every five or ten years, by computing a_n as of that date. Also, each item in Net Amortization multiplied by $1 + i$ gives the next item. The logarithmic solution is difficult, but needs to be clearly understood, as the situation is rather frequent. Problem: Find the present worth of a 15-year annuity of \$1.00 at 4¾% annually.

The compound discount must first be found.

$1 + i = 1.0475$ nl		.0201540316
times 15		.302310474
colog		9.697689526—10
$\ln = v^n$	=	.4985282
from 1	=	.5014718 compound discount
divided by .0475 = 10.5573		

Effective rates: The changes in the formulae when j_p must be used are the same as in the case of the amount of an annuity. The formulae are:

For use with tables, a_n times $\dfrac{i}{j_p}$

For use with logarithms, $\dfrac{1 - v^n}{j_p}$

In both cases n is the number of years; but v^n is reckoned at rate i.

The significance of the quantity a_n is twofold. In the first place it is the present worth of a debt of 1 due periodically for n periods. It is the present payment which is equivalent to such a debt and will extinguish it with all interest that will become due on the successive dates of payment.

Another significance is this: It is the present investment which will allow the owner to withdraw 1 periodically; in this case the interest due periodically is added to the fund remaining in the hands of the trustee or banker, and serves to lengthen the time required to consume the investment.

Three special types of annuity are common:

An annuity due is an annuity payable at the beginning of the period. Insurance premiums and rentals were mentioned as common instances. For such an annuity the amount and present worth are indicated by the symbols s_n and a_n. Suppose an annuity due of $1.00 annually for five years at 4% annually.

We will first find the amount of this annuity:

First	payment,	accumulated	5	periods,	$1.21665290
Second	"	"	4	"	1.16985856
Third	"	"	3	"	1.124864
Fourth	"	"	2	"	1.0816
Fifth	"	"	1	"	1.04
Total					$5.63297546

From the tables we find the amount of an ordinary annuity for **six** periods to be 6.63297546. This differs from the amount of the annuity due for five periods by one cash instalment. The reason is obvious. The ordinary annuity for six periods ends with a cash payment on which no interest is earned. The previous payments have earned interest during one to five periods. The annuity due is exactly the same excepting for the cash payment.

Amount of an annuity due, first rule:

Find the amount of an ordinary annuity for one more period, and subtract one cash payment.

Another aspect of the case: Each payment of the annuity due has been improved one period more than each payment of an ordinary annuity for the same time. So if we take the amount of an ordinary annuity for the same time and multiply by $1 + i$ we shall have the amount of an annuity due. $5.41632256 \times 1.04 = 5.63297546$.

Formulae for these two rules:

First, $\mathbf{s}_n = s_{n+1} - 1$

Second: $\mathbf{s}_n = s_n$ times $1 + i$

The present worth of an annuity due of 1 per period for five periods at 4% annually is:

First	payment, cash				$1.00
Second	"	discounted	1	period	.96153846
Third	"	"	2	"	.92455621
Fourth	"	"	3	"	.88899636
Fifth	"	"	4	"	.85480419
					$4.62989522

Now the present worth of an ordinary annuity for **four** years is $3.62989522, which differs from that of the annuity due by one cash payment. This is because each payment of the ordinary annuity for four years is discounted from one to four periods. The annuity due for five periods has an extra cash payment, made when the annuity was entered on, and this payment is not discounted. So one way to find the present worth of an annuity due is to find the present worth of an ordinary annuity for one less period, and add one cash payment. This is the first rule. Another rule corresponding to the second rule for summation of an annuity due is this: Multiply the present worth of an ordinary annuity for the same time by $1 + i$. $4.55182233 \times 1.04 = 4.62989522$. This is evidently because an annuity due is payable one period nearer the time when it is entered on, and so each payment is subject to discount for one less period than the payments of an annuity made at the end of the period.

Expressed as formulae, these rules are,

First, $a_n = a_{n-1} + 1$

Second, $a_n = a_n$ times $1 + i$.

A deferred annuity is one which is not entered until after a certain time. The amount of such an annuity is not different from the amount of an ordinary annuity. No payments are made until after the period of deferment and consequently there are no extra interest accumulations.

The present worth, however, is important is some transactions. There are two methods of finding the present worth of such an annuity.

Problem: Find the present worth of a five-year annuity of 1 per year, deferred three years, on a 4% annual basis. At the date on which the annuity is entered, its present worth is \$4.45182233. This amount must be discounted to the present time. Multiplying by v^4 at 5% we find $4.45182233 \times .88899636 = \$3.95765384$.

Rule: Multiply the present worth of the annuity by the present worth of 1 for the period of deferment.

Another method, useful when table values are obtainable, is the following:

a_8	at 4%	=	\$6.73274487
a_3	at 4%	=	2.77509103
Subtracting,			\$3.95765384

Rule: Subtract the present worth of an annuity for the period of deferment from the present worth of an annuity for the total time from the present to the last payment.

Formulae for these two rules are:

For an n-period annuity deferred m periods:

First, a_n times v^n.

Second, $a_{n+m} - a_m$.

A perpetuity is a series of periodic payments which are expected to continue indefinitely. The dividends on a share of preferred stock are one instance; the income from a farm is another. The amount of such an annuity has no meaning, but the present worth is important in some cases, such as an endowment of a charitable institution, or the financing of permanent improvements.

By one of the laws of simple interest, the principal can be found if we know the periodic income and rate: income divided by rate equals principal. If a perpetuity of 1 annually is valued on a 5% annual basis, its present worth is $20.00. The symbol for a perpetual series or infinite length of time is $_{m}$.

Rule: Divide the rent by the periodic rate of interest:

As a formula, $a_{\infty} = \frac{1.}{i}$

The interest rate on a given annuity can be found approximately by Baily's formula. The only method of finding the exact rate is to assume a rate and find the sum or present worth as the case may be, and continue estimating and computing until the correct rate is found. Baily's formula always gives a rate slightly above the correct, but it is accurate enough for most occasions. It is necessary to know the periodic payment, the term, and either the amount or the present worth. The first step is to read from the tables the nearest value of s_n or a_n for the given time. Indicate these by s' and a'. Indicate the interest rate for s' or a' by j.

Baily's formula provides for finding a corrective which we will call q. Then the correct rate $i = j + q$ or $i = j - q$. The formulae for q are

$$q = \frac{j\,(s' - s)}{s' - n\,(1 + j)^{n-1}}$$

$$q = \frac{j\,(a' - a)}{a' - nv^{n+1}}$$

Required the income rate on an annuity of 1 yearly for 10 years, when the present worth is 8. In the table of a_n in the 10-year row we find 7.91271818 in the 4½% column. v^{11} at 4½% is .61619874.

$$q = \frac{.045(8 - 7.91271818)}{7.91271818 - (10 \times .61619874)}$$

which gives q = .00224. Since the value of a′ chosen was less than a, the rate must have been too high. Subtracting .00224 from .045 gives i = .04276.

PROBLEMS ON CHAPTER VII

Find the amount of these ordinary annuities of $1.00 each:

73. 12 years at 4½% annually.
74. 20 years at 5% nominal quarterly.
    (That is, $.25 quarterly.)
75. 8 years at 5% effective quarterly.
76. 9 years at 4½% nominal quarterly.

Find the present worth of these ordinary annuities of $1.00 each:

77. 8 years at 5% annually.
78. 11 years at 6% nominal semiannually.
79. 14 years at 5% effective semiannually.
80. 6 years at 5½% annually.
81. 24 years at 4¼% nominal semiannually.

Find the amount of these annuities due of $1.00 each:

82. 6 years at 5% nominal semiannually.

83. 8 years at $4\frac{1}{2}\%$ nominal quarterly.
84. Find the present worth of an annuity due of $1.00 a year for 15 years at $4\frac{1}{2}\%$ annually.
85. Find the present worth of a 30-year annuity of $1.00 on a $4\frac{1}{2}\%$ annual basis, deferred 10 years. Solve two ways.
86. If $50.00 is deposited at the end of each six months for 5 years in a bank paying 4% nominal semi-annually, what will be the amount credited to the depositor at the end of the term. Set up a schedule.
87. A man buys a house for $1,000.00 cash and $1,000.00 at the end of each year for three years. On a 6% annual basis what is the cash equivalent of the purchase price?
88. The present worth of an ordinary annuity of $1.00 a year for 8 years on a 5% annual basis is given in the tables as $6.4632. Prove by setting up a schedule.
89. A man on his 40th birthday wishes to purchase a 10-year annuity of $1,000.00 a year, the first payment to be made on his 65th birthday. The insurance company bases on 3% annually. Omitting probability and loading, what is the net single premium? (If the first payment is made on his 65th birthday, the annuity is entered on the 64th birthday. Remember also that insurance premiums are prepaid).
90. A concern wishes to move their manufacturing plant from its present location on leased property. The lease has ten years to run, and the rent is payable annually in advance. The annual rental is $15,000.00 a year for the first five years and $16,000.00 a year for the last five years. What is the cash value of the lease on a 5% annual basis.
91. The sum of a 15-year annuity of $1.00 a year is $18.00. What is the annual interest rate?

92. A debt of $5,000.00 is to be paid in 20 annual payments of $400.00 each. What is the interest rate? (First reduce this to a unit basis. The present worth of this annuity is $12.50 per dollar of annual payment.)

CHAPTER VIII

SINKING FUNDS

At the beginning of Chapter VII we discussed the amount that could be accumulated by periodic payments of 1 improved at a given rate of interest. In that case we knew the periodic payment and computed the amount. A more common type of problem is that in which we know the amount which is to be accumulated and the time and rate, and must find the periodic payment. Such an amount to be accumulated within a given time is called a sinking fund, and the periodic payment is called the sinking fund payment. Such a sinking fund payment can be found by division from the table of s_n. If annual payments of 1 will amount in five years at 4% annually, to 5.41632256, what is the annual payment necessary to amount to 1 in five years at 4% annually? By the principle of division which was mentioned some time ago— that if a product and one factor are known, the other factor can be found by division—the sinking fund payment to amount to 1 is evidently 1 divided by 5.41632256, or .18462711.

In general, if periodic payments of 1 accumulated for n periods amount to s_n, then to accumulate 1 the periodic payment must be 1 divided by s_n, or, expresssed as a fraction, $\frac{1}{s_n}$

Tables of this quantity are given in most collections. In some collections, however, tables of another quantity, $\frac{1}{a_n}$

are given. In such cases, as will be explained in Chapter X, $\frac{1}{s_n}$ can be found by subtracting i from the table value of $\frac{1}{a_n}$

Since $\frac{1}{s_n}$ is the reciprocal of s_n, the formula must be the reciprocal of the formula for s_n. The formula is

$$\frac{1}{s_n} = \frac{i}{s^n - 1} \text{ or } = \frac{i}{(1 + i)^n - 1}$$

This is for use in logarithmic computation.

Rule: Divide the single interest by the compound interest.

The following problem and schedule will illustrate the principles of sinking fund mathematics:

Problem: What is the annual sinking fund contribution necessary to provide a sum at the end of five years to retire \$10,000.00 bonds, on a 4% annual basis. The table value is .18462711; multiplied by 10,000 gives the annual payment, \$1,846.2711.

This amount can be proved. The amount of five annual payments of 1 improved at 4% annually is given in the tables as 5.41632256; and multiplied by the amount of the annual payment, \$1,846.2711, the product is about 10,000.

When such a sinking fund is begun it is usual and advisable to build a schedule covering the entire life of the fund. The schedule for the above bonds is:

Schedule for Sinking Fund to Redeem $10,000.00 5% Bonds, Accumulated at 4% Annually

Year	Interest at 4%	Annual Instal-ment	Total Accumula-tion	Amount in Fund
1		$1,846.2711	$1,846.2711	$1,846.2711
2	$ 73,8508	1,846.2712	1,920.1220	3,766.3931
3	150.6557	1,846.2811	1,996.9268	5,763.3199
4	230.5328	1,846.2712	2,076.8040	7,840.1239
5	313.6050	1,846.2711	2,159.8761	10,000.0000
Totals	768.6443	9,231.3557	10,000.0000	

Note that on any date the total of Annual Instalments plus Interest is the total of Total Additions and also the Amount in the Fund as of that date. Also that each Total Addition multiplied by 1 + i gives the next Total Addition. If the schedule runs for a long time, the amount in the fund ought to be proved just as in the other schedules which have been given. We have shown that the amount in the fund on any date can be computed in advance by multiplying the sinking fund factor by the amount of an annuity for the given number of periods. In the above case the amount at the end of the third year will be $1,846.2711 multiplied by s_n for three years: $1,846.2711 × 3.1216 = $5,763.3198.

This bond problem is unusual in that we have assumed annual interest payments, whereas the usual bond pays interest semiannually. The general subject of bond redemption will be reserved for the chapter on bonds, at which time the method of handling semiannual interest payments will be fully discussed.

The computation of the sinking fund payment for effective rates, unusual rates which are computed by logarithms, and annuities due will now be considered. The situations are in general the exact reverse of those treated in the chapter on s_n.

Effective rates: The formula is

$$R = \frac{1}{s_n} \div \frac{i}{j_p}$$

Rule: Divide the sinking fund payment for the time in years by the effective rate factor.

Problem: What is the monthly payment required to accumulate \$1,000.00 in 5 years at 5% effective, convertible monthly?

.1809748 ÷ 1.0227148 = .1769563 the yearly charge per dollar.

.1769563 × 1,000 ÷ 12 = \$14.7463.

Problem involving logarithms:

The formula has already been explained:

$$R = \frac{i}{s_n - 1}$$

Problem: What is the semiannual payment into a sinking fund to accumulate \$200,000.00 in 25 years at 4¼% nominal, convertible semiannually.

1 + i	=	1.02125 nl	.0091320695
times 50			.456603475
ln	= s^n =		2.8615623
less 1			1.8615623
divided into .02125			.01141514
times 200,000	=		2283.028

Annuity due: The formula is

$$R = \frac{1}{s_n} \div (1 + i)$$

Rule: Divide the sinking fund factor for an ordinary annuity by the compound interest ratio.

Problem: What payment beginning now and continuing through five years from date will accumulate \$1,000.00 at 4½% annually.

This is an annuity due of six payments.

$\frac{1}{s_6}$ is given in the tables as	.14887839
divided by 1.045 =	.1424673
times 1,000 =	\$142.4673

To determine the time required to accumulate a stated amount: let K be the amount and R the periodic payment. The formula is

$$n = \frac{\log\left[1 + \frac{Ki}{R}\right]}{\log\,(1 + i)}$$

Rule: Multiply the amount to be accumulated by the periodic rate and divide by the rent; add the result to 1 and find the log. Divide this log by the log of the compound interest ratio. (Six-place log is sufficient.)

Problem: In what time can a fund of \$10,000.00 be accumulated by semiannual payments of \$500.00 improved at 5% nominal, semiannually?

K =	10,000
i =	.025
product	250
divided by 500	.5
add 1	1.5
nl	.176091
1 + i = 1.025 nl	.010724
quotient	16.42 semiannual periods.

PROBLEMS ON CHAPTER VIII

93. What is the semiannual payment into a sinking fund which is to be accumulated at 5% nominal, convertible semiannually, to redeem \$50,000.00 of bonds at maturity, 25 years from date?

From "Mathematical Theory of Investment," by Professor Ernest Brown Skinner.

94. A debt of \$250,000.00 is to be paid in one sum at the end of 20 years. The sinking fund is to be accumulated at 4% annually. How large is the fund at the end of ten years? (Schedule is not necessary.)
95. What sum must be deposited quarterly in a bank which pays 4% nominal quarterly to amount to \$1,000.00 in 10 years?
96. In what time will \$50.00 deposited semiannually in a bank which pays 4% nominal semiannually amount to \$1,000.00?
97. What sum paid at the end of each month to an insurance company which accumulates at 4% effective will amount to \$1,000.00 in 10 years?
98. What would be the monthly payment in problem 97 if made in advance?
99. What will be the annual payment into a sinking fund accumulated at 4% annually to extinguish a debt of \$225,000.00 at the end of 25 years. Build a schedule for the first ten years and prove the amount in the fund at the end of the tenth year.

CHAPTER IX

VALUATION OF ASSETS

One of the commonest purposes for the building up of a sinking fund is the replacement of an asset. Such a fund is called a reserve fund for depreciation. When an asset is purchased, its probable effective life and replacement cost must be estimated. The phrase effective life is used because in any modern plant an asset is scrapped long before it has reached the limit of its working power. This is due to obsolescence, the loss of effectiveness in the face of more modern methods and machines. This aspect of the replacement problem must be kept in mind when the term of the sinking fund is set, so that the reserve may be sufficient in amount at a fairly early date. Also the replacement cost will be much higher than the original cost; how much higher is usually difficult to estimate. It is customary also to set an arbitrary value on the wornout machine as scrap. The cost, less scrap value, if any, is called the wearing value; and it is this wearing value which must be accumulated during the effective life of the machine.

The reserve fund then is a sinking fund which must be accumulated during the life of the machine. The amount of the fund is replacement cost less scrap value, and the term is the effective life of the machine. The replacement cost is indicated by C; the scrap value by S; the life by n; and the wearing value by W. $W = C - S$, and the formula for the annual depreciation charge is

$$D = W \times \frac{1}{s_{n}}$$

Rule: Multiply the wearing value by the sinking fund factor for the given time and rate.

Problem: Determine the annual depreciation charge against an asset which will cost $1,000.00 to replace less a scrap allowance of $50.00 at the end of 5 years, on a 3½% annual basis.

$$W = \$950.00$$
$$n = 5$$
$$\frac{1}{s_{n}} = .18648137$$
$$D = \$177.1573$$

A schedule should be set up at the time the asset is acquired; one form is as follows:

Year	Interest on Fund	Annual Payment	Total Addition	Total Fund
1		$177.1573	$177.1573	$177.1573
2	$ 6.2005	177.1573	183.3578	360.5151
3	12.6181	177.1573	189.7754	550.2905
4	19.2602	177.1573	196.4175	746.7080
5	26.1347	177.1573	203.2920	950.0000
	$ 64.2135	$885.7865	$950.0000	

Notice that the total Annual Charge plus the total Interest is the Wearing Value. Also that the Total Addition for any year multiplied by $1 + i$ gives the Total Addition for the next year. From the schedule the periodic Journal entry is made, and the amount in the reserve at any time can be read from the Total Fund column. This amount can be verified at any time by multiplying the annual charge by s_{n}. The total of the above fund at the end of the third year is $177.1573 × 3.106225 = $550.2903.

Valuation of a plant as a whole: In a manufacturing plant the wearing values and terms of effective life of the machines vary widely, yet it is important to arrive at some method of considering the plant as a whole. The depreciation charge on each machine or class of machines must be computed separately, and the results combined, as in this problem:

Problem: On a 4% annual basis find the annual charge for depreciation against the following electrical power plant:

Part	Life	Wearing Value	Annual Charge
Building	40	$ 8,000.00	$ 84.1879
Engine	20	3,500.00	117.5361
Boiler	16	2,000.00	91,6400
Dynamo	18	7,000.00	272.9533
		$20,500.00	$566.3173

The depreciation charge against a plant as a whole can conveniently be indicated by D′.

Starting with this annual charge on the whole plant, four important factors can be worked out:

The rate of depreciation, d, is the ratio of the periodic charge to the wearing value at the beginning; for this plant it is 566.3173 ÷ 20,500 = 2.7625%.

The composite life of a plant is the time within which the total of the additions to the depreciation fund or funds will accumulate to the original wearing values. This is exactly the time problem of page 79:

$$n = \frac{\log\left[1 + \frac{i}{d}\right]}{\log(1 + i)}$$

Rule: Divide i by the rate of depreciation, add one, and find the log; divide by the log of 1 + i.

This notion is important when a bond issue is secured by a mortgage on a plant and equipment. Such bonds should never be issued for term longer than the composite life of the plant, and to provide a proper margin of safety the time should be considerably shorter. For the above plant

$\frac{i}{d}$	=	1.448
add 1 and find log		.388811
1.04	nl	.017033
quotient		22.83 years

The wearing value of an asset at any time is the original wearing value less the amount in the reserve at that time.

The condition percent of a plant at any time is the ratio of the wearing value at that time to the original wearing value.

Other methods of reckoning depreciation:

By allowing interest on the wearing value of the asset at the time. The formula is

$$D = (Cs^{n} - S) \times \frac{1}{s_{n}}$$

Rule: Multiply the cost by the compound amount factor, and subtract the scrap value; multiply the result by the sinking fund factor.

The composition of this formula is easily understood. The original cost multiplied by the compound amount shows the capital tied up in the machine during its entire life. Of this a small part will be returned as a scrap allowance. Deducting scrap, we have a loss of capital which must be spread over the life of the machine by the sinking fund method. The objection to this method is that interest ought to be reckoned on the original cost if it is to be reckoned at all. For the asset considered at the beginning of this chapter, the annual charge by this method is

1,000 × 1.18768631 less 50 × .18648137 = $212.1573

The schedule is

Year	Net Book Value	Interest 3½%	Annual Charge	Reduction in Book Value
1	$1,000.00	$35,0000	$212.1573	$177.1573
2	822.8427	28.7995	212.1573	183.3578
3	639.4849	22.3819	212.1573	189.7754
4	449.7095	15.7398	212.1573	196.4175
5	253.2920	8.8653	212.1573	203.2920
	$50.0000	$110.7865	$1,060.7865	$950.0000

The phrase net book value indicates the difference between the cost and the amount in the reserve. It is difficult to make proper Journal entries from such a schedule because the asset should always be carried on the books at cost. Yet the interest must be reckoned on the net book value, else this method of charging for depreciation cannot be used.

Fixed percentage on a diminishing book value: The formula is

$$r = 1 - \sqrt[n]{\frac{S}{C}}$$

Rule: Subtract the log of C from the log of S; divide the result by n; find the antilog and subtract from 1. For the asset we have been considering

S	nl	1.698970
C	nl	3.000000
subtracting		8.698970—10
divided by 5		9.739794—10
ln		.54928
from 1		.45072
or		45.072%

each year of the net book value at the beginning of that year. The schedule is

Year	Net Book Value	Depreciation 45.072%	Amount in Reserve
1	$1,000.00	$450.72	$450.72
2	549.28	247.5714	698.2914
3	301.7086	135.9861	834.2775
4	165.7225	74.6944	908.9719
5	91.0281	41.0281	950.0000
	$50.0000	$950.0000	$3,842.2608

Hatfield, in his "Modern Accounting," discusses these two methods in detail. He says that the best arguments for this last method are, first that the depreciation charge ought to be heaviest when the machine is new and less in need of repairs, so that year by year as repair charges increase they may be compensated by lessening depreciation charges; and second, that investigations show that a machine depreciates most rapidly in its first year and that the rate of actual physical depreciation decreases after that time.

Any reader who is interested in the more complicated methods of reckoning depreciation, such as the equal annual payment method and the unit cost method, should read Saliers' "Principles of Depreciation", in which they are treated fully, with schedules and diagrams.

Miscellaneous formulae covering certain problems in valuation and accounting for assets:

Provision against the total exhaustion of wasting assets such as mines, timber lands, oil properties, and so on. If such assets were permanent or renewable, the provision would take the form of a perpetuity, which is valued by the formula $\frac{R}{i}$ when R is the annual income.

But since the asset has a limited life it is necessary to provide also against the time when the rent will cease. Such provision should take the form of a sinking fund payment. This payment, at a rate which we may indicate by i′, must be added to i as an extra expense. The formula for the value of the asset is then

$$V = \frac{R}{\frac{1}{s_n} \text{ (at rate } i') + i}$$

Rule: Add the sinking fund charge at rate i′ to the income rate i; divide the result into the periodic rent.

Problem: Find the value of a mine which will net \$15,000.00 a year for 30 years, if the annual dividend rate is to be 6% payable annually, and the sinking fund is to be accumulated at 3½% annually.

$\frac{1}{s}$ 30 years at 3½%	=	.01937133
i	=	.06
adding		.07937133
dividing into 15,000	=	\$188,985.116

Viewed from another standpoint this solution gives the amount of stock which can safely be issued. This can be proved. Basing on 1,900 shares of stock at 100,

annual dividends, 6% on \$190,000.00....	\$11,400.00
sinking fund, \$188,985.116 times .01937133..................	3,660.897
	\$15,060.897

which is slightly more than the annual income estimated at \$15,000.00. Of course it would be conservative to allow for contingencies by issuing a smaller amount of stock.

Capitalization of assets: This is defined as the first cost plus cost of perpetual renewals. The cost of a perpetuity is $\frac{1}{i}$; the rent of a sinking fund is $\frac{1}{s_n}$. So the product of these two factors is the cost of an indefinite series of renewals occurring every n years. The capitalized cost is the first cost plus this cost of indefinite renewals. This problem occurs especially when endowments for hospitals, colleges, and so on have to be valued. There are usually items of upkeep which occur every year, and these must be valued as perpetuities.

Problem: A philanthropist desires to present a building to a university. The building will cost \$200,000.00 and will have a life of 50 years. The annual upkeep will cost not over \$16,000.00. What must be the amount of his gift?

First cost........................	\$200,000.00
Renewals $200{,}000 \times \frac{1}{.04} \times .0065502$	32751.00
Upkeep $\frac{16000}{.04}$	400,000.00
	\$632,751.00

It is necessary to distinguish carefully between

$\frac{1}{i}$ which is the value of a series of annual payments;

and $\frac{1}{i} \times \frac{1}{s_n}$ which is the value of a series of payments at intervals of n years.

SUMMARY OF FORMULAE USED IN ACCOUNTING FOR ASSETS

Let C be the cost, original or replacement;
S the estimated scrap value;
W the wearing value;
n the life of a single asset;
n′ the composite life of an entire plant;
D the depreciation charge on a single asset;
D′ the depreciation charge on an entire plant;
d the rate of depreciation of an entire plant, being the ratio of the total periodic charge to the wearing value at the begining.

Annual depreciation $D = W \times \frac{1}{s_n}$

Amount in fund at end of r-th year $S_r = D \times s_n$.

Rate of depreciation $d = \frac{D}{W}$

Composite life of a plant $n' = \frac{\log \left[1 + \frac{i}{d}\right]}{\log (1 + i)}$

Wearing value at end of r-th period $W_r = W - D \times s_n$.

Condition percent $= \frac{W_r}{W}$

Allowing interest on book value $D = (C \times s^{n} - S) \times \frac{1}{s_{n}}$

Fixed rate on diminishing book value $r = 1 - \sqrt[n]{\frac{C}{S}}$

Value of a wasting asset to pay dividends at rate i′ when R is the annual output:

$$V = \frac{R}{\frac{1}{s_{n}}\text{(at rate } i'\text{)} + i}$$

Value of a perpetuity, payable annually $= \frac{R}{i}$

Value of a perpetuity, payable every n years $= \frac{R}{i} \times \frac{1}{s_{n}}$

Capitalized cost of an asset (this will be treated in the next chapter on amortization, but the formula is given here for the sake of completeness):

$$X = C \times \frac{1}{a_{n}} \times a_{n+m}$$

For most of these formulae I am indebted to The Mathematical Theory of Investment, by Professor Ernest Brown Skinner, of the University of Wisconsin.

PROBLEMS ON CHAPTER IX

100. An asset having an estimated life of 10 years and a scrap value of at least \$500.00 will cost \$12,000.00 to replace. Set up a schedule of the reserve by the sinking fund method, basing on 3½% annually.

101. An asset having an estimated life of at least 8 years will cost \$8,000.00 to replace less an allowance of \$200.00 on the old machine. Set up a schedule of the reserve on the fixed percentage method.

102. An asset having an estimated life of 5 years and estimated replacement cost of \$15,000.00, less scrap

allowance of \$1,200.00 is to be valued by the method of allowing interest on the book value at the beginning of each year. Set up a schedule on a 4% annual basis.

103. In an investigation of a public utility, the various parts were appraised as follows:

Part	Life	Cost	Scrap
A	15	\$150,000.00	\$ 8,000.00
B	25	65,000.00	3,800.00
C	45	122,000.00	6,750.00
D	30	85,500.00	15,625.00
E	8	8,300.00	650.00

On a 3% annual basis compute

a) annual depreciation charge, sinking fund method;

b) rate of depreciation;

c) composite life.

104. What is the wearing value and condition percent at the end of the 7th year of the first asset of problem 103?

105. Certain timber lands will yield a net revenue of \$8,000.00 a year for 35 years. 6½% stock, dividends annual, is to be issued, and provision is to be made for annual payments to a sinking fund at 4% annually. Supposing that the land has no residual value, what is a reasonable purchase price?

106. It is estimated that a mine will yield \$12,000.00 per year and will be exhausted in 20 years. The stock is to pay 7% annually and the sinking fund is to be accumulated at 4½% annually. Find the amount of stock that can be safely issued.

107. How much can a railroad afford to spend in eliminating a grade crossing which is guarded by two

watchmen at \$650.00 a year each. Assume 4% annually.

108. What sum must be set aside annually to provide for rebuilding every 25 years of a bridge costing \$30,000.00? Assume that money will be worth 4% annually.

109. On a certain railroad grade a helper engine and two crews are necessary at an annual cost of \$12,000.00; a new engine must be provided every 20 years at a cost of \$14,000.00. How much can the railroad afford to spend to level the grade if money will be worth 4½% steadily?

110. A man desires to endow an art gallery. The building will cost \$75,000.00. The life is 100 years. The annual cost for purchase and rental of paintings and so forth will be \$50,000.00. Repairs and renewals will cost \$2,000.00 yearly. The lighting system will need renovating every 5 years at a cost of \$1,200.00. Upkeep and service will cost \$4,500.00 a year. Value the endowment on a 3½% annual basis.

CHAPTER X

AMORTIZATION

In Chapter VII we discussed amortization somewhat. It can be defined as the series of equal payments which will extinguish a given debt, the periodic payments being used first to meet interest accrued on balances outstanding. In that chapter we learned that five equal payments of 1 will extinguish a debt of 4.45182233 with interest at 4%. The more common situation is that in which the amount of the debt is known and the amount of the periodic payment must be found. What, for instance, is the periodic payment which will extinguish a debt of 1 in 5 years, interest on outstandings at 4% annually? Evidently, if payments of 1 will extinguish a debt of 4.45182233, then the payment required to extinguish a debt of 1 is 1 divided by 4.45182233, or .22462711. And the payment required to extinguish a debt of $10,000.00 is $2,246.2711. A schedule will prove this:

Schedule of Amortization of a Debt of $10,000.00 in 5 Equal Annual Instalments, with Interest on Outstandings at 4% Annually

Year	Balance Outstanding	Interest at 4%	Annual Payment	Net Amortization
1	$10,000.0000	$400.0000	$2,246.2711	$1,846.2711
2	8,153.7289	326.1492	2,246.2711	1,920.1219
3	6,233.6070	249.3443	2,246.2711	1,996.9268
4	4,236.6802	169.4672	2,246.1712	2,076.8040
5	2,159.8762	86.3950	2,246.2712	2,159.8762
		1,231.3557	11,231.3557	10,000.0000

The usual checks should be observed. Total Annual Payments less total Interest is total Net Amortization, which is the same as the Balance Outstanding at the beginning. Each Net Amortization times $1 + i$ gives the next amount. The balance outstanding on any interest date is the present worth of the annual payments still due. In the above schedule, the amount due just after the third payment is the present worth of the two remaining payments; $2,246.2711 × 1.88609467 = $4,236.678. If the schedule is lengthy it should be proved every few years with this last check.

To pass to algebra, since a debt of a_n can be amortized by a series of n payments of 1, a debt of 1 can be amortized by a series of n payments of 1 divided by a_n, or, expressed as a fraction, $\frac{1}{a_n}$.

For logarithmic computation, the formula is the reciprocal of the formula for a_n,

$$\frac{1}{a_n} = \frac{i}{1 - v^n} = \frac{i}{1 - \frac{1}{(1+i)^n}}$$

and in the usual circumstances, when the given rate is effective, if we are using table values, we divide: $\frac{a_n}{1} \div \frac{i}{j_p}$ where n is the number of years; and if we are using logarithms, we use j_p instead of i in the numerator of the formula, while n in the denominator is the number of years. These will be illustrated later.

Now, if we compare the interest columns of the sinking fund and amortization schedules, we shall find an interesting relation between the sinking fund and amortization factors. In the sinking fund schedule the interest is a credit; in the amortization schedule it is a debit.

Year	Amortization Debit	Sinking Fund Credit
1	$400.00	
2	326.1492	$ 73.8508
3	249.3443	150.6557
4	169.4672	230.5328
5	86.3950	313.6050

In each year the sum of the two items is 4% of $10,000.00. Or, a debit of $400.00 set against each credit to the sinking fund results in the corresponding debit to amortization.

This shows why the tables do not include any table of $\frac{1}{a_n}$; for that quantity is always greater than $\frac{1}{s_n}$ by the annual interest. This fact, stated on the basis of a unit, as usual, gives the formula:

$$\frac{1}{a_n} = \frac{1}{s_n} + i$$

Rule: To find the amortization factor for any time and rate, read the table value for the corresponding sinking fund factor and add the interest rate.

In the sinking fund table for 4% when n = 5 we find .18462711; adding .04 we have the amortization factor .22462711, the same amount we found on the previous page by division.

Logarithmic solution: Find the rent of an amortization of a debt of 1 in 10 semiannual payments at 5½% nominal, convertible semiannually:

1 + i = 1.0275 nl	.0117818305
times 10	.117818305
colog = log of v^n	9.882181695—10
antilog = v^n	.7623977
from 1	.2376023 the compound discount
divide into .0275 =	.11574

Rule: Divide the interest rate by the compound discount.

Effective rates: What is the monthly payment required to amortize a debt of \$500.00, beginning at the end of one month and continuing for 5 years, interest at 5% effective.

$$\frac{1}{a_{\overline{n}|}} = .2309748$$

$$\frac{i}{j_p} \text{ at } 5\%,\ p = 12, = 1.0227148$$

quotient = .2258448

times 500 = 112.9224

divided by 12 = \$9.41

Annuity due: Since the formula for the present worth of an annuity due is $a_{\overline{n}|} \times (1 + i)$, it is evident that the amortization factor payable in advance is $\frac{1}{a_{\overline{n}|}} \div (1 + i)$.

Problem: A debt of 1 is to be amortized in 5 annual payments, the first payment to be made immediately. What is the annual payment on a 6% annual basis?

$$\frac{1}{a_{\overline{5}|}} = .2373964$$

dividing by 1.06 = .223959

Four special problems in amortization are important:

The amount due after any number of payments have been made: This has been mentioned already, and was explained as the present worth of the payments still due. This is true immediately after a payment has been made. If, however, the amount due immediately before a date of payment is in question, the payments still due constitute an annuity due and must be valued according to the formula for such an annuity.

Problem: A debt of \$10,000.00 is to be amortized over a period of 10 years by annual payments, interest at 5%

annually. On the date of the seventh payment it is desired to liquidate the balance in one amount. What is the balance?

$\frac{1}{a_{10}}$	=	.12950458
times 10,000		1,295.0458 annual payment
a_3		2.72324803
product		3,526.7309
add 7th payment		1,295.0458
Amount due		$4,921.7767

Or, we might have multiplied $3.549505 \times 1.05 \times 1{,}295.0458$, since 3.5459505 is the present worth of an ordinary annuity for 4 periods at 5%.

The time required to amortize a given debt: Let A be the debt and R the rent, or periodic payment. The formula is*

$$n = \frac{\operatorname{colog}\left[1 - \frac{Ai}{R}\right]}{\log(1 + i)}$$

Rule: Multiply the amount by the interest rate, divide by the rent. subtract from 1, find the colog; divide by the log of the interest ratio.

Problem: In what time can a debt of $1,000.00 be liquidated by annual payments of $80.00, interest at 5% annually?

A	=	$1,000.00
R	=	80.00
i	=	.05
Ai	=	50
divided by 80		.625
from 1		.375

From "Mathematical Theory of Investment," by Professor Ernest Brown Skinner.

nl		9.574031—10
colog		.425969
1.05	nl	.021189
quotient		20.1

When the rate is effective, j_p is used in place of i in the numerator (multiplied by A), but R is the annual payment, and i is the annual rate.

Problem: In what time will a debt of $300.00 be amortized by quarterly payments of $10.00, beginning 3 months from date, interest at 3½% effective?

A	=	$300.00
R	=	40.00
j_p when p = 4		.0345498
A times j_p		10.36494
divided by 40		.2591235
from 1		.7408765
nl		9.869746—10
colog		.130254
1.035 nl		.01494
quotient		8.72
or		8½ years and a small balance due.

The increased cost of lengthening the life of an asset

Let C be the cost of the present asset, and C′ the cost of the longer-lived asset. Let n be the present life, and m the added life, so that n + m is the total life of the new asset. Now, if we multiply the present cost by the amortization factor, we shall have the annual cost of the present asset. And if we multiply that by the present worth of an annuity of 1 for the lengthened term of the new asset we shall have the amount which we can afford to pay for the new asset. Expressed as a formula*

*From "Mathematical Theory of Investment," by Professor Ernest Brown Skinner.

$$C' = C \times \frac{1}{a_n} \times a_{n+m}$$

Problem: If a piece of equipment costs $100.00 and has a life of 3 years, how much can be paid for a better piece of equipment which will last 5 years, on a 4% annual basis?

C	=	$100.00
$\frac{1}{a_3}$	=	.36034854
product		36.034854 annual cost
a_5	=	4.45182233
product		$160.42

That is, the equipment costs us .36034854 a year, for any term of years. The present worth of a 5-year article is that amount times the present worth of an annuity of 1 for 5 years.

The amortization of a series of bonds will be discussed in the next chapter.

PROBLEMS ON CHAPTER X

111. A man buys an estate valued at $8,000.00, agreeing to pay cash $2,000.00 and the balance in 5 equal annual payments with interest on outstandings at 5% annually. Set up a schedule of the payments.

112. A man borrows $3,000.00 at 5% annually, agreeing to pay $300.00 annually. By the use of a formula determine the life of the debt. What is the amount due after the last full period. Solve without schedule.

113. What is the annual rent to amortize a debt of $1,000.00 in 10 years at 4¾% annually.

114. What quarterly payment continued three years will pay a debt of $200.00 with interest at 4% effective.

115. What is the annuity required to amortize a debt of \$1,500.00 with interest at 3½% annually, in 5 payments, the first payment to be made at once.

116. Set up a schedule for problem 115.

117. A debt of \$3,000.00 with interest at 4½% annually is to be amortized in 10 annual payments, beginning one year from date. If the payments are made promptly for five years, what amount will liquidate the balance due on the date of the sixth payment.

118. A city incurs a debt of \$150,000.00. From a mathematical standpoint which is better: to amortise in 20 equal annual payments at 6¼% annually, or to accumulate a sinking fund by annual payments improved at 4% annually, paying interest on the face of the debt meanwhile at 5% annually.

119. A man is offered \$12,000.00 a year payable in advance on a 99-year lease or cash \$225,000.00 for a piece of land. Which is better for him on a 5% annual basis?

120. On a 4½% annual basis what amount can be spent in improving a piece of apparatus which now costs \$25.00 and has a life of 3 years so that it may last 5 years?

121. On a 5% annual basis what can a company afford to spend in improving a machine which now costs \$85.00 and has a life of 6 years so that it may last 9 years?

CHAPTER XI

VALUATION OF BONDS

A bond is a promise to pay a certain sum, usually \$1,000.00 or \$100.00, after a certain number of years from the date of issue. The commonest exception to this statement is the issue of a series redeemable by drawings. It also promises, during the life of the bond, to pay interest at a fixed rate, called the coupon rate; such interest is usually paid semiannually. In this book this coupon rate will be indicated by c; actuarial writers usually use j, because, as we shall see, this coupon rate is really a nominal rate and does not indicate the income rate realized by the holder. Bonds are commonly purchased either above par, "at a premium"; or below par, "at a discount". The amount of premium or discount may depend on a number of things; but if the bond is "gilt-edged" and purchased for investment rather than for speculation the price depends chiefly on the income rate expected, and after that on the value at which the bond will be redeemed and the time to elapse before redemption. The income rate, which we will call i, is independent of c; in fact it is usually not calculated until after purchase and when the bond is brought on the books. If the coupon and income rates are the same the bond is bought at par. If the purchaser is satisfied that his income shall be less than the nominal coupon rate, the bond will be purchased at a premium. If the purchaser wishes his income rate to be above the coupon rate, he will purchase at a discount.

The value of a bond depends then on the redemption value and the relation between the coupon and income rates. Since coupons are usually payable semiannually it is customary to value bonds on a semiannual basis, unless the bond is stated to pay annual or quarterly coupons. Let us value a bond by valuing a) the present value of the redemption price, and b) the present value of the coupons.

Problem: What is the price of a 7% 25-year bond, redeemable at par $1,000.00, to yield 6%.

Since the valuation is to be on a semiannual basis:

n	=	50
c	=	.035
i	=	.03

Present worth of the redemption value is $1,000 \times v^{50}$ at 3%, which is $1,000 \times .22821708 =$	$228.1071
Present worth of 50 coupons of $35 each is $35 \times 25.729764 =$	900.5417
Total	$1,128.6488

This establishes the correct value of the bond. The best method of valuation is the formula of Mr. Makeham, as follows:

Let C be the redemption value (cash);
K the present worth of C;
c the coupon rate (Makeham uses j);
i the income rate;
A the present value of the bond.

The formula is $A = K - \frac{c}{i} \times (C - K)$

Rule: Find the present worth of the redemption value. Subtract this from the redemption value and multiply the result by the coupon rate and divide by the income rate. Add these results.

For the bond we have just been considering:

$$K = 1{,}000 \times .22810708 \quad = \quad \$228.1071$$

$$C - K = 771.8929$$

$$\frac{.035}{.03} \times 771.8929 \quad = \quad \underline{900.5417}$$

$$\$1{,}128.6488$$

The reason for this formula is this. If the bond were purchased so that coupon and yield rates were the same, the price would be the same as the redemption value. Since the present worth of the redemption value is $228.1071, the value of the coupons would be the remainder of the $1,000.00, or $771.8929. But the yield rate is lower than the coupon rate, which shows that the bond cost more than par. It cost as much above par as the coupon rate is above the yield rate, which in this case is $\frac{.035}{.03}$. In every case the coupon rate is the numerator of this fraction.

It is usual to set up a schedule at the time the bond is brought on the books; the first ten periods of this schedule are:

Schedule of Amortization of 7% 25-Year Bond of the..Co.,
Redeemable at Par $1,000.00 on..............,
Income Rate 6%

Period	Coupons 3½%	Income 3%	Net Amortization	Book Value
				$1,128.6488
1	$35.00	$33.8595	$1.1405	1,127.5083
2	35.00	33.8252	1.1748	1,126.3335
3	35.00	33.7900	1.2100	1,125.1235
4	35.00	33.7537	1.2463	1,123.8772

5	35.00	33.7163	1.2837	1,122.5935
6	35.00	33.6778	1.3222	1,121.2713
7	35.00	33.6381	1.3619	1,119.9094
8	35.00	33.5973	1.4027	1,118.5067
9	35.00	33.5552	1.4448	1,117.0619
10	35.00	33.5119	1.4881	1,115.5738
	350.00	336.9250	13.0750	

There are several checks to such a schedule, the first of which is by the usual additions. If the bond was bought at a premium, total Coupons less total Income equals total Net Amortization; each Net Amortization multiplied by $1 + i$ gives the next. If the bond was bought at a discount, total Income less total Coupons gives total Net Accumulation; and each Net Accumulation times $1 + i$ gives the next.

The next check is by valuing the bond again:

$$K = 1{,}000 \times v^{40} =$$

$$1{,}000 \times .3065568 \quad = \quad \$306.5568$$

$$C-K = 693.4432$$

$$\frac{.035}{.03} \times 693.4432 \quad = \quad 809.0171$$

$$\$1{,}115.5739$$

Still another check is by differencing. This has the merit that it is not in any way similar to the methods used in valuation, and so is less likely to repeat any mistake that may have been made. Set down in a column the successive Net Amortizations (in the table they have been set down in reverse order to facilitate subtraction). Subtract each one from the one directly over it; this is called the first order of differences. Repeat the operation, securing a second order of differences. If desirable subtract once more. In the accompanying schedule this is unnecessary. It is unnecessary to proceed beyond the point where the differences are nearly equal, some being slightly more and some slightly less. If any wide variation should appear it would indicate an error in that region of the schedule.

1.4881	.0433	.0012
1.4448	.0421	.0013
1.4027	.0408	.0011
1.3619	.0397	.0012
1.3222	.0385	.0011
1.2837	.0374	.0011
1.2463	.0363	.0011
1.2100	.0352	.0009
1.1748	.0343	
1.1405		

In the second column the differences are progressive; that is, they decrease successively. In the third column the differences waver. In a short-term schedule the differences are larger than in a long-term schedule. If this were a five-year bond it would probably be necessary to obtain a fourth column to reach the point where the differences were nearly equal.

When the income rate is not in the tables the use of logarithms is necessary in finding v^n. The process has been demonstrated, but will be shown here again.

Problem: Value a 5% 20-year bond redeemable at par $1,000.00, to yield 3.6%.

c	=	.025
i	=	.018
n	=	40
1 + i = 1.018	nl	.0077477778
times 40		.30991112
colog		9.69008888—10
ln = v^n =		.4898794

$$K = 1{,}000 \times .4898794 \qquad = \quad \$489.8794$$

$$C-K = 510.1206$$

$$\frac{2.5}{1.8} \times 510.1206 \qquad = \quad \underline{708.5008}$$

$$\$1{,}198.3802$$

Redemption at a premium or discount: Bonds are often issued subject to an agreement that they may be redeemed above or below par. This is especially common when the issuing company reserves the right to redeem before maturity. Suppose a 5% bond can be redeemed at 110, when par is 100. The income rate is based on the purchase price, so the redemption value has no direct effect on the income rate. But 5% on 100 is not 5% on 110. The coupon rate must be converted to this new value at maturity. $\frac{100}{110} \times .05 = .045454$ annually, or .022727 semiannually. Suppose this bond is redeemable in 15 years, that is n = 30. Income rate 4%:

$$K = 1{,}100 \times v^{30} \text{ at } 2\%$$

$$= 1{,}100 \times .55207089 \qquad = \quad \$607.2780$$

$$C-K = 492.7220$$

$$\frac{.022727}{.02} \times 492.7220 \qquad = \quad \underline{559.9113}$$

$$\$1{,}167.1893$$

Valuation with allowance for income tax: The writers on actuarial science are forced to treat this subject at length and to give complicated formulae because the English income tax law bases taxation on the net income after accumulation of discount or amortization of premium. But in this country where the tax is levied on the coupons themselves, the problem is simple. Consider a 20-year 5% bond redeemable at par $1,000.00. The purchaser wishes to net 4% after taxation, and estimates that the tax will consume 10% of all his income. Of every 2.5% of income one-tenth will satisfy the tax. That means that he will net 2.25%. The problem then is to value a 40-period 2¼% bond on a 2% yield:

$$K = 1{,}000 \times .4528904 \qquad = \qquad \$452.8904$$

$$C-K = 547.1096$$

$$\frac{.0225}{.02} \times 547.1096 \qquad = \qquad \underline{615.4983}$$

$$\$1{,}068.3887$$

Without the allowance for tax the cost of the bond would have been $1,136.7774, the difference being 68.3887.

The proof of this is very interesting. The holder of this bond expects to pay $2.50 of every coupon in tax. The present worth of a 40-period annuity on a 2% basis is 27.35547924. The product of 27.35547924 × 2.50 is $68.3887, which was exactly the decrease in purchase price due to the allowance for taxation.

Valuation between interest dates: So far we have been valuing bonds on interest dates or at least ex-interest. In practise, bonds are usually bought between interest dates, and if the price is accurately fixed there must be not only an allowance for interest due the former holder but also adjustment for discount or premium for the part of a period already past. It is usual to reckon the interest and premium or discount by simple interest, as follows:

Problem: A 7% 25-year bond, redeemable at par $1,000.00, is bought on a 6% basis. The coupons are due January and July 1. The bond is bought March 1. What is the price?

We found that the price January 1 was $1,128.6488. The seller is entitled to the third of the coupon which has accrued and must expect to lose the third of the premium to be amortized:

Value January 1		\$1,128.6488
1/3 of \$35.00	\$11.6667	
less 1/3 of 3% of book value	11.2865	.3802
Subtracting, value March 1		\$1,128.2686

Also, it often happens that the coupon dates do not coincide with the dates for closing the books. In this case it is desirable to write a schedule which will show the value of the bond on the dates of closing.

Problem: A bond redeemable for \$1,000.00 on October 15, 1916, bears interest at 5% payable April and October 15. The bond is purchased April 15, 1913, to yield 6%. Value the bond and write a schedule which can be used as a source of entries on a set of books which are closed June 30 and December 31:

Value April 15		\$968.8486
5/12 of 3% of book value	\$12.1106	
5/12 of coupon	10.4167	1.6939
Adding, value June 30		\$970.5425

The schedule is:

Date	Income 3%	Coupons	Net Accumulation	Book Value
4/15/'13				\$968.8486
6/30/'13	\$12.1106	\$10.4167	\$1.6939	970.5425
12/31/'13	29.1163	25.00	4.1163	974.6588
6/30/'14	29.2398	25.00	4.2398	978.8986
12/31/'14	29.3670	25.00	4.3670	983.2656
6/30/'15	29.4980	25.00	4.4980	987.7636
12/31/'15	29.6329	25.00	4.6329	992.3965
6/30/'16	29.7719	25.00	4.7719	997.1684
10/15/'16	17.4149	14.5833	2.8316	1,000.0000
	\$206.1514	\$175.0000	\$31.1514	

The usual checks should be observed. A slight discrepancy will usually appear in the net accumulation or amortization for the last partial period. In the above schedule this has all been taken up by adjusting the income. There are various methods of eliminating this residue, as it is called. It can all be taken care of during the first period or during the last period. It can be spread equally over all the periods. Or it can be spread over all the periods proportionately to the net accumulations or amortizations. By this last method an adjustment per dollar is found, basing on the total of the net column. This adjustment is then multiplied by each accumulation in turn, and the result is added to or subtracted from the proper item. For a detailed discussion with tables see the Sprague-Perrine "Accountancy of Investment". A last method which may be suggested is to multiply the error by $\frac{1}{s_{\overline{n}}}$ at rate i, and add the result to the Income items or subtract from them as may be necessary.

Serial bonds: Bonds are often issued under an agreement that they may be redeemed at various dates rather than all at once. These periodic redemptions may take place semiannually, annually, or at intervals of several years. The bonds may be redeemable at par or at a premium or discount; in practise they are often redeemable at a premium which decreases as the final redemption date approaches. They may be redeemable at the pleasure of the trustee of the sinking fund or in stated amounts, regular or otherwise.

If the trustee is instructed merely to purchase bonds in open market to the amount of the fund in his hands at any time, it is obviously impossible to value the series with any mathematical accuracy. For such redemptions will vary with varying market conditions, both price and amount being variables.

If the schedule of redemptions is known, dates, amounts and prices being stated, the series can be valued as a whole. No single bond can ever be valued except at a so-called average price, which is only an arbitrary valuation, unless the redemption date of that particular bond is stated. The valuation of an irregular series proceeds as follows:

Problem: A series of $100,000.00 of bonds paying 5% annually is issued on January 1, 1920, redeemable on and after January 1, 1926, in these amounts:

Jan. 1, 1926	$10,000.00
Jan. 1, 1927	15,000.00
Jan. 1, 1928	20,000.00
Jan. 1, 1929	20,000.00
Jan. 1, 1930	35,000.00

Value this series on a 4% annual basis.

It is first necessary to schedule the interest payments:

Jan. 1, 1921, to Jan. 1, 1926	$5,000.00 each year
Jan. 1, 1927	4,500.00
Jan. 1, 1928	3,750.00
Jan. 1, 1929	2,750.00
Jan. 1, 1930	1,750.00

The value of the series on a 4% annual basis is:

Jan. 1, 1921, to Jan. 1, 1925, $5{,}000 \times a_5$	$22,259.1117
Jan. 1, 1926, $15{,}000 \times v^6$	11,854.7180
Jan. 1, 1927, $19{,}500 \times v^7$	14,818.3973
Jan. 1, 1928, $23{,}750 \times v^8$	17,353.8924
Jan. 1, 1929, $22{,}750 \times v^9$	15,983.8483
Jan. 1, 1930, $36{,}750 \times v^{10}$	24,826.9832
Total	$107,096.8909

If a series is regular in every way, redeemable in equal amounts at a fixed price and at equal intervals of time, it can be valued by Makeham's formula. Consider a series of ten $1,000.00 bonds, 5% JJ, redeemable one each six months beginning July 1, 1922, at par. This series is to be valued as of Jan. 1, 1917, ex interest on a 6% basis. The method of stating the coupon rate is a common method, and means that the bonds pay 2½% on the first of January and July.

This is a 10-period series deferred 10 periods. So in Makeham's formula the present worth factor, instead of v^n, is the factor for a deferred annuity, which is $a_{n+m} - a_m$, where m is the term of deferment. In this case it is $a_{20} - a_{10}$

C	$=$	$10,000.00
$K = 1{,}000 \times a_{20} - a_{10}$ at 3%		
or		6,347.2720
$C - K = 3{,}652.7280$		
$\frac{.025}{.03} \times 3{,}652.7280$	$=$	3,043.9400
		$9,391.2120

The problem is usually complicated by the fact that redemptions are annual, while interest payments are semiannual. In such a case the yield is also semiannual, and the best method is to annualize both coupon and income rates. A series of five $10,000.00 bonds, 5% JJ, issued Jan. 1, 1920, is redeemable at par, $10,000.00 at the end of each year for five years. Value the series on a 4½% basis.

The rates can be changed to an annual effective basis by the rule on page 35.

$(1.025)^2$	$=$	1.050625
annualized c	$=$	5.0625%
$(1.0225)^2$	$=$	1.0455
annualized i	$=$	4.55%

Next we have to find a_5 at 4.55%

1.0455	nl	.01932398745
times 5		.09661993725
colog		9.90338006275—10
$\ln = v^n$		.80053466
from 1		.19946534
divided by .0455	=	$4.3838536 = a_n$

Now, using Makeham's formula

$K = 10{,}000 \times a_n$	=	\$43,838.54
$C-K = 6161.464$		
$\frac{5.0625}{4.55} \times 6161.464$	=	6,855.47
		\$50,694.01

If the bonds are to be redeemed by an amortization scheme the valuation is no more difficult. Consider a series of five \$10,000.00 bonds, paying 5% annually, to be amortized over a period of ten years, the first amortization to take place at the end of one year. To value the series on a 4% annual basis.

$\frac{1}{a_{10}}$ at 5%	=	.12950548
$50{,}000 \times .12950548$	=	\$6,475.229

This is the annual amortization payment, disregarding the necessity of redeeming whole bonds. If this annuity is purchased on a 4% annual basis,

a_{10} at 4%	=	8.11089578
times 6,475.229	=	\$52,519.9075

Redemption of a series: As has been said, a series may be redeemed at the discretion of the trustee, or the arrangements as to amounts and dates may be stated in the deed of trust under which the bonds were issued. Two methods which lead to some mathematical difficulties will now be discussed.

It may be desirable to retire the series by **an amortization scheme,** so that periodic payments on account of principal and coupons combined may be as nearly equal as possible. Of course the amount of principal repaid at each redemption date must be an exact multiple of the face of the bonds. In such a case the periodic payments cannot be exactly equal, and the schedule requires considerable adjustment. Also this situation is not to be confused with the sinking fund method, by which no bonds are redeemed until the entire series matures, and periodic payments are allowed to accumulate interest in the hands of the trustee.

Suppose an issue of 50 bonds of par $1,000.00 is to be retired over a period of ten years, beginning one year from date. Interest on bonds outstanding is to be at the rate of 5% annually. The amortization factor is .12950458, which, multiplied by 50,000, gives the average annual cost $6,475.229. The first year's coupons are $2,500.00, leaving $3,975.229 for redemptions. This will almost redeem four bonds, and it will be good policy to appropriate the necessary $24.77. The annual payment then is $6,500.00, and the par of bonds outstanding at the beginning of the second year is $46,000.00. 5% interest for the second year is $2,300.00, leaving $4,175.229 for redemptions. It will be best to appropriate only $4,000.00. The extra $175.229 may be appropriated if desired and left to draw interest and later it can be used to make up deficiencies such as occurred in the first year. This is ultra conservative, however. The schedule can now be given without comment:

Schedule of Amortization of $50,000.00 Bonds at Par $1,000.00, with Interest at 5% Annually, in Ten Equal Annual Instalments

Year	Par of Bonds Out-standing	Interest 5%	Total Payment	Net Payment	Number of Bonds Retired
1	$50,000.00	$2,500.00	$6,500.00	$4,000.00	4
2	46,000.00	2,300.00	6,300.00	4,000.00	4
3	42,000.00	2,100.00	6,100.00	4,000.00	4
4	38,000.00	1,900.00	6,900.00	5,000.00	5
5	33,000.00	1,650.00	6,650.00	5,000.00	5
6	28,000.00	1,400.00	6,400.00	5,000.00	5
7	23,000.00	1,150.00	6,150.00	5,000.00	5
8	18,000.00	900.00	6,900.00	6,000.00	6
9	12,000.00	600.00	6,600.00	6,000.00	6
10	6,000.00	300.00	6,300.00	6,000.00	6
		14,800.00	64,800.00	50,000.00	50

The usual checks should be observed. If there were a question regarding the payment on account of redemptions in any year, it would be wise to redeem too few bonds, rather than too many. This satisfies the bondholders and allows the issuing company the use of the money for one year more. In general the number of bonds redeemed is progressive, increasing slowly from the first date to the last.

The other method which we ought to consider is the method which requires what we may term a redemption fund. By this method annual costs are equalized regardless of amount of redemptions. This is done by appropriating a fixed amount and allowing any excess to remain in the bank until it is needed. In this way interest is earned on all unused balances, yet they are available without any strain on the finances of the issuing company.

A series of $100,000.00 of 5% bonds such as we valued on page 110 is a good example. Suppose we appoint a trustee and agree to pay him equal amounts at intervals of six months from six months after the date of issue to the date of the last redemption. This arrangement does away with the necessity of annualizing rates and simplifies the problem. The trustee is to meet all coupons and redemptions and keep the balance of the fund invested at a rate which we assume will be not less than 4% annually. The balance thus invested must be available in any year when total redemptions and coupons exceed the annual payment into the fund.

First it is necessary to value the series at the date of issue. This value has already been found to be $107,096.8909. The annual payment to meet this must be 107,096.9809 times $\frac{1}{a_{10}}$ at 4%, which is .12329094; the product is the annual payment, $13,204.0763. The schedule is given on the following page.

Instalment bonds: Sometimes a bond is payable, principal and interest, at regular intervals and in equal amounts. This is the usual amortization problem. Consider a bond for $5,000.00 to be repaid in ten semiannual instalments with interest on outstandings at 4% nominal, convertible semiannually, payments to be equal in amount. The bond is to be priced on a 3% basis. The periodic payment is 5,000 × .11132653 = $556.6327.
The present worth of these ten payments at 1½% is 556.6327 × 9.22218455 = $5133.3695.
The two operations may be expressed in one formula:

$$A = C\left(\frac{1}{a_n} \text{ at rate c}\right) \times (a_n \text{ at rate i}).$$

Determining the income rate on a bond: This is a subject into which the accountant need not go as deeply as the actuarial writers, but a few of the simpler methods may be useful at times.

For most purposes the income rate can be read accurately enough from bond tables. If the rate obtained from the tables is not acceptable, a closer rate may be obtained by interpolation between two table values. Selecting the nearest values above and below the given value proceed as in finding logarithms. A $1,000.00 bond redeemable at par after 40 years, bearing interest at 4%, is bought at 93.50, that is, $935.00. In the tables for a 40-year 4% bond we find

	yield	4.35	cost	$942.955
		4.30		952.116
differences		.05		9.161
partial differences		x		2.116

$$x : .05 = 2.116 : 9.161$$
$$x = .1155$$
$$\text{rate} = 4.26155\%$$

Schedule of Payments Into 4% Redemption Fund for $10,000.00 5% Bonds, and Retirement of the Bonds with Interest

Year	Amount in Fund, Beginning of Year	Interest 4%	Additional Payment	Total at End of Year	Payments on Account of Interest	Payments on Account of Redemptions
1	....	....	$13,204.0763	$13,204.0763	$5,000.00	....
2	$8,204.0963	$328.1629	13,204.0763	21,736.3355	5,000.00	....
3	16,766.3355	669.4534	13,204.0763	30,609.8652	5,000.00	....
4	25,609.8730	1,024.3949	13,204.0763	39,838.3442	5,000.00	....
5	34,838.3442	1,393.5338	13,204.0763	49,435.9543	5,000.00	....
6	44,435.9543	1,777.4382	13,204.0763	59,417.4688	5,000.00	$10,000.00
7	44,417.4688	1,776.6988	13,204.0763	59,398.2439	4,500.00	15,000.00
8	39,898.2439	1,595.9298	13,204.0763	54,698.2500	3,750.00	20,000.00
9	30,948.2500	1,237.9300	13,204.0763	45,390.2563	2,750.00	20,000.00
10	22,640.2563	905.6103	13,204.0763	36,749.9429	1,750.00	35,000.00
Totals	$267,728.8223	$10,709.1521	$132,040.7630	$410,478.7374	$42,750.00	$100,000.00

Note the various checks. Total of Amount in Fund at Beginning plus total Interest at 4% plus total Additional Payments equals total of Total at End of Year. Total Interest plus total Additional Payments equals total of the two right-hand columns. The error of $.0571 is negligible. If it were large enough to need adjustment, it could be spread over the amounts in the Additional Payments column in any one of the ways mentioned on page 109.

After finding table values it is possible to proceed by the method of trial and error. That is, in the bond just discussed, after finding the 4.26155%, and ascertaining that the corresponding price is $949.99, the yield rate can be slightly increased and the bond valued logarithmically. Of course this will eventually give as correct a value as the number of places used can offer. It is very laborious, and generally unnecessary. It should be borne in mind that too high a yield rate results in too low a price; and too low a yield rate results in too high a price.

The next method is given by Sprague, and is perhaps the best of all, in that the result of each step checks up the assumed rate. It is based on a $100.00 bond, and may be expressed in the following rule:

1) Assume an income rate; half of this will be i;
2) find $a_{\overline{n}|}$ for $1.00 at rate i;
3) take half the result of 2);
4) divide the result of 3) into the known premium or discount;
5) if the bond was bought at a premium, subtract the result of 4) from c; if bought at a discount, add;
6) compare the result of 5) with 1); modify 1) and repeat until the result of 5) is near enough to 1) to be acceptable.

Problem: To determine the income rate on a 5% bond bought at 110, redeemable at par after 20 years. The successive steps may be:

Trial Rate	$a_{\overline{n}|}$	Resulting Rate
4	27.35548	4.268885
4.1	27.11714	4.262459
4.2	26.88179	4.25341
4.25	26.76526	4.252763
4.252	26.76254	4.252687
4.2524	26.75967	4.25261

The income rate is apparently about 4.2525. Valuing the bond at that rate we find $1,100.0134.

Two of the many methods given by Todhunter are simple and fairly accurate if used with care.

First, Let k be the premium per dollar of redemption value; that is $k = \frac{A-C}{C}$; then

$$i = \frac{c - \frac{k}{n}}{1 + \frac{n+1}{2n} \times k}$$

Problem: Determine the income rate on a 4½% bond redeemable at $1,125.00 after 25 years, bought at 120. Adjusting c, the periodic coupon rate is 2%.

A	=	$1,200.00
C	=	1,125.00
A—C	=	75.00
divided by C	=	.06666——this is k
divided by 50	=	.0013333——
from c		.0186666——

This is the numerator of the formula.

n + 1	=	51
2n	=	100
quotient		.51
times k		.034
add 1		1.036

This is the denominator.

The quotient is	1.8053 semiannually,
or	3.6106% nominal.

The correct rate is 3.5936%, an error of .017 annually. If the bond is bought at a premium this result will be too high; if bought at a discount the result will be too low.

Second: Assume a rate i′ and compute K′ as for Makeham's formula. If K′ is not near enough to the given A, modify the rate i′. This part of the solution can be shortened by use of bond tables. After a satisfactory K′ has been found, use this formula:

$$i = c + i' \left[\frac{C-A}{C-K'}\right]$$

For the bond just considered the solution is:

Assume i′		=	1.8% semiannually.
To find K′ at this rate:			
1.018		nl	.007747778
times 50			.3873889
colog			9.6126111—10
ln =	v^n	=	.409837
times 1,125		=	461.0666 this is K′

$$i = 2 + 1.8\left[\frac{1125-1200}{1125-461.0666}\right]$$

Note that the numerator of the parenthesis is negative, so that the value of the fraction must be subtracted from 2.

i	=	1.796664%
or		3.5933% annually.

PROBLEMS ON CHAPTER XI

Value these bonds (rates c and i are quoted for the whole year):

122.	par $1,000.00	c 4%	i 3%	n 40 years
123.	1,000.00	3½%	2½%	5 build a schedule
124.	1,000.00	5%	3.6%	15
125.	1,000.00	4%	5%	45
126.	1,000.00	4½%	3½%	33½
127.	1,000.00	4%	3%	12

128.	1,000.00	7%	5%	10 redeemable at 108
129.	1,000.00	3½%	4¾%	20 redeemable at 105
130.	1,000.00	5% JJ	3%	due Jan. 1, 1918; purchased April 1, 1915
131.	1,000.00	7%	5%	interest June 15 and Dec. 15, due June 15, 1916, purchased Oct. 1, 1898
132.	1,000.00	5%	6%	20 income tax 15%
133.	1,000.00	4½%	5%	30 redeemable at 102; income tax 25%
134.	1,000.00	3½%	3%	20 income tax 5%

135. A 4% bond AO, redeemable at $1,000.00 on Oct. 1, 1916, is purchased April 1, 1914, on a 6% basis. Value the bond and set up a schedule to be used as the source of entries on a set of books which are closed June 30 and December 31.

136. Find the value on a 4% basis of a series of bonds amounting to $100,000.00 issued on January 1, 1885, paying 5% JJ, redeemable $2,000.00 each six months at par on and after July 1, 1910.

137. A series of bonds amounting to $50,000.00, bearing 5½% annually is to be redeemed
 $8,000.00 at the end of one year at 105,
 9,000.00 at the end of two years at 103,
 10,000.00 at the end of three years at 102,
 11,000.00 at the end of four years at 101,
 12,000.00 at the end of five years at 100.
 Value the series on a 4½% annual basis.

138. A series of bonds to the value of $250,000.00 bearing 5% annually is to be amortized over a period of ten years. Value the series on a 6% annual basis.

139. Set up a schedule for the company issuing the above bonds showing how the annual payments on account of coupons and redemptions combined may be kept as nearly equal as possible. Redemptions will be at par $1,000.00.

140. A series of bonds to the amount of $500,000.00, bearing 6% annually, is issued January 1, 1908. Par of the bonds is $1,000.00. For 10 years they pay interest only, and beginning January 1, 1919, the series is to be amortized over a period of 20 years. What must be the annual payment into a redemption fund which can be accumulated at 4½% annually for the full thirty years from January 1, 1908, to January 1, 1938, to meet the interest charges and retire the bonds?

141. A series of bonds to the value of $50,000.00 is issued under an agreement that they shall be redeemed $10,000.00 each year for five years. They pay 6% JJ, and are purchased on a 5% annual basis. What is the price?

Find the income rate on the following bonds; if possible use the method given by Sprague (see page 118), and check by one of the methods taken from Todhunter, or else by valuing the bonds logarithmically. Bond tables may be used for first approximations, but answers must be accurate to hundredths of a percent.

142. A 5% 50-year bond, redeemable at par, bought at 110.

143. A 4% 20-year bond, costing 115, redeemable at 108.

144. A 5% 20-year bond is quoted at "95 to yield about 5.4%." Is this accurate? If not, quote a more accurate yield rate.

CHAPTER XII

THE SLIDE RULE

This chapter is a brief explanation of only such uses of the slide rule as are useful in general accounting practise. The real study of the slide rule must be in the form of practise; simple problems should be solved by arithmetic and the solutions checked by the slide rule. Only in this way can proficiency be attained.

The slide rule is a mechanical device for performing multiplication, division, raising to a power (by continued multiplication), square roots, and in general the easier operations which can be performed by the use of logarithms.

Among the commercial operations for which the slide rule is used are payrolls, invoices, discounts, setting prices on goods to be sold, inventories, determining rates of profit and loss, proportion, foreign prices and exchange, simple interest and discount.

The mechanical and engineering uses are numerous, and valuable to the cost accountant. Such processes should be taught in engineering courses, however, and will not be mentioned here.

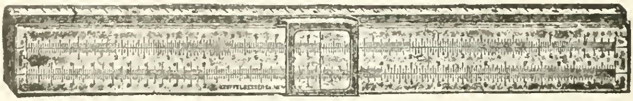
There are many sizes and forms of calculating machines built on the logarithmic principle. Beside the usual eight and ten-inch slide rules there are twenty-inch rules for desk use. Special rules have trigonometric scales, scales of equal parts for use with logarithmic tables, cube root scales, and so on. Other machines are circular or cylindrical, thus securing greater accuracy by reason of their longer scales.

Our discussion will concern the shorter slide rules, which are most useful to the accountant. Computations with the ten-inch rule are accurate to the third digit, and the fourth place can be estimated with a fair degree of accuracy when the result is at the left end of the scales. Results to more than four places are useful only to check actual computation.

The logarithmic scale: The logarithms of the numbers from 1 to 10 are:

1	nl	0
2		.301030
3		.477121
4		.602060
5		.698970
6		.778151
7		.845098
8		.903090
9		.954243
10		1

If we multiply each of these logarithms by ten and lay the lengths off in sequence on a line ten inches long we have this scale:



The usefulness of such a scale consists in the fact that as multiplication can be performed by adding logarithms, so multiplication can be performed by adding lines whose logarithmic measures correspond to the given numbers. For instance, the sum of the logarithms of 2 and 3 is .778151, which is the logarithm of 6. If we have two lines whose lengths are respectively the logarithms of 2 and 3, the sum of the lengths must be the logarithm of 6. This statement can be tested from the above scale with the edge of a sheet of paper or any other straight edge, whether it has any scale marked on it or not.

A **slide rule** is composed of three parts: stock, slide, and runner. The runner has a line or wire on its glass, to assist in locating or keeping the location of points. On the face of the rule are four scales, lettered A, B, C and D; A and D are on the stock, B and C are on the slide. A and B are alike, being scaled from 1 to 10 twice; C and D are scaled only once. In other words, A and B are "double scales".

Consider first the D scale; it is precisely like the scale just constructed from the logarithms, with the addition of subdivisions. These subdivisions are not on a logarithmic scale, but are equal divisions of the section of the scale in which they occur. This corresponds to the manner in which we interpolate in the use of logarithms, by ordinary proportion. Of course such results are not strictly accurate, but they are satisfactory for ordinary use. The meaning of these subdivisions is evident. If we suppose that the 1 at the left is 10, then the subdivision next it marked 1 is 11; and the subdivision next after that is 12. In the division beginning with 2, the subdivision marked 1 is 21, and so on. The subdivision which we called 11 is marked off into ten very small parts; the first of these is 111 with the decimal point wherever the problem requires; and so on. If the 1 at the extreme left were 1000, then the subdivision 1 would be 1100, and the sub-subdivision 1 would be 1110.

The scale begins at 1 because the logarithm of 1 is 0, and zero length is at the beginning of the scale. This 1 may be 1 or 100 or 1,000,000 or .01; the logarithms of all these numbers differ only in their characteristics, and the slide rule is a scale of mantissae and not of characteristics. The characteristic in any result can usually be determined by common sense although there are rules which will be given later.

The A scale differs from the D scale only in that the logarithmic scale is repeated and the subdivisions are less numerous and shorter. If the 1 at the extreme left is 1, then the 1 in the middle is 10; if the 1 at the left is 100, the 1 in the middle is 1000. If the 1 at the left is .01, the 1 in the middle is .10; and so forth. Any number on the A scale is the square of the number directly below it on the D scale. For instance, move the runner until the wire is over 9 (not 90) on the A scale; directly under the wire on the D scale find 3.

The rules for determining the characteristic are:

In multiplication:

If the slide projects to the right, the characteristic is the sum of the characteristics of the factors plus 1;

If the slide projects to the left, sum of the characteristics.

In division:

If the slide projects to the right, the characteristic is the characteristic of the dividend less that of the divisor;

If the slide projects to the left, this difference less 1.

A few simple illustrations:

Multiplication: $3 \times 4 = 12$.

On C and D; set 1 of C to 3 of D; below 4 on C read 12 on D. Notice that in order to get 4 of C over the D scale at all, the slide must be moved to the left. The characteristic is determined by the first clause of the rule for multiplication.

On A and B; set 1 of B to 3 (not 30) of A; on A over 4 of B read 12. Note that the slide projects to the right, but the characteristic is 1 as before; this is because the result is in the right hand half of the double scale A.

Division: $35 \div 5 = 7$.

On C and D; set the runner to 35 of D; bring 5 of C to the runner; below 1 of C read 7 on D. The difference of the characteristics is 1; but the slide projects to the left, so the characteristic of the quotient is 1 less than that difference. See the last part of the rule for the characteristic in division.

On A and B: set the 5 of B to 35 of A; over 1 of B read 7 on A.

Notice that the **divisor is on the slide**, and is set opposite the dividend on the stock.

To square a number: $7^2 = 49$.

Set the runner to 7 of D; on A under the runner read 49.

Square root: to find the square root of 81.

Set the runner to 81 of A; under the runner on D read 9.

Proportion: Most engineering problems involve the use of "gauge points", which are ratios comparing two quantities; for instance, the diameter and circumference of a circle; the speed of two shafts; and such ratios. Commercial gauge points are the ratio of a yard to a metre, foreign exchange rates, and many others. There is always a list of such gauge points as are constant on the back of a slide rule. A few illustrations:

What number of yards is equivalent to 350 metres? The gauge point is 75:82; that is, 75 metres are equivalent to 82 yards. Set 75 of C to 82 of D; under 350 of C read 382 on D.

What is the value in francs of $125.00, exchange at 5.26. Proceed as in multiplication: 125 × 5.26 = 657.5.

At the same quotation what is the value in dollars of 2,000 francs? This is division. Set 5.26 of C to 2,000 of D; under 1 of C read 3803. The characteristic is 3 — 0 — 1 = 2; answer $380.30. Correct value, $380.25. An article costing $2.75 is to be sold to make 75% on the cost. Find the selling price.

The selling price is 175% of the cost. Multiply 2.75 by 1.75, product $4.82.

An article costing $3.25 is to be sold to gain 55% on the selling price. Find the selling price. Evidently the cost is 45% of the selling price, and the solution requires division. Set .45 of C to 3.25 of D; under 1 of C read $7.22.

Sometimes this use of gauge points leads to a puzzling situation.

Problem: Find the weight of 10 gallons of water. The gauge point is 3:25; that is, 3 gallons weigh 25 pounds. Set 3 of C to 25 of D; the answer is expected under 105 of C, but this point is outside the stock. The slide must be thrown to the right its whole length; to assure accuracy, set the runner to the one at the right end of the slide, and then throw the slide over until the one at its left end is under the runner. This multiplies the result by 10 Under 105 of C read 87.5.

Successive operations: What is the interest on a $50.00 Liberty Bond at 3½% from November 15 to May 15? The time is 181 days. Expressed as cancellation, the problem is:

$$\frac{50 \times .035 \times 181}{365}$$

Set 1 of C to 50 of D; runner to .035 of C, and under it is one year's interest (characteristic 0). Set 1 of C to this result, runner to 181 of C (characteristic 2). Set 365 of C to the runner, and under 1 of C read 87. (Characteristic 2 − 2 − 1 = −1). Answer .87.

PROBLEMS ON CHAPTER XII

A. Drill problems to train in fundamentals:

36×14 $\quad$ 82×74

$875 \div 35$ $\quad$ $4588 \div 74$

$28 \times 36 \div 63$

$36 \times \dfrac{7}{3}$

23^2 $\quad$ 23^3 $\quad$ 23^4

$\sqrt[2]{1681}$ $\quad$ $\sqrt[3]{15376}$

B. Commercial problems:

145. What amount is due to an employee who has worked 46½ hours at $18.00 for a 48-hour week?
146. What amount is due this employee if he works 3¼ hours' overtime, at "time and a half?"
147. A chair is listed at $16.50 less 10—10—5%. What is the net cost?
148. Name a selling price on the above chair to gain 75% on the cost.
149. A 60-day note for $318.00 is dated May 9 with interest at 5½% exact time. What is the value at maturity?
150. The above note is discounted May 11 at 6%, bankers' time. Find the net proceeds.
151. What is the proceeds of a Russian bill for 550 rubles, exchange at .12¾?

152. What is the proceeds of a draft on London for £385 4s., exchange at 4.855?

153. What draft on London can be purchased for $1,250.00, exchange at 4.8675?

154. When gold is quoted at $20.60 per ounce, what is the price in francs per gramme, exchange at 5.75? Gauge point, ounces to grammes 6:170.

155. What is the adjusted coupon rate on a 3½% bond redeemable at 108?

156. In taking an inventory in a hardware store 38 clamps were counted. They were billed from the manufacturer at $6.50 per gross less 25—10—10%. The dealer priced them to gain 50% on net cost. What is the inventory value, depreciation excluded? (Value one clamp and multiply by 38.)

157. A desk costing $85.00 is to be priced to gain 40% on the selling price. State the selling price.

158. A sprinter ran 100 metres in 11 seconds. What is the equivalent speed for 100 yards?

159. An article costing $25.00 is sold at $40.00 less 10—10%. What is the percent of profit on cost?

160. The financial report of a business showed the following facts, in thousands of dollars:

Sales		315
Cost of Sales	189	
Selling Expenses	68	
General Expenses	22	
Profit	?	

What percent of Sales is each of the four items? Total must be approximately 100%.

CHAPTER XIII

REVIEW PROBLEMS

161. What is the cash balance October 1, at 5% exact time, of the following account:

    Debits: May 8, \$500.00, 2 months; June 25, \$1,000.00, 60 days; August 10, \$500.00, cash.

    Credits: July 1, \$1,200.00, 30 days; September 15, \$200.00, cash.

162. When a boy was born \$500.00 was placed to his credit in a bank which pays 5% nominal, compounding semiannually. If the account is not disturbed until his twenty-first birthday, what will be the balance on that date?

163. A man buys a farm for \$10,000.00, paying cash \$2,500.00 and agreeing to pay the balance with accrued interest in three equal annual instalments, interest at 6% annually. What is the annual payment? Prove by a schedule.

164. A clerk expects to go into business for himself as soon as he has saved \$5,000.00. If he has now \$2,500.00, and can invest all funds at 4½% annually, how much must he save yearly so that he may have the necessary amount at the end of five years?

165. How many years will be necessary to accumulate \$1,000.00 by investing \$150.00 per annum at 6% annually? Solve without schedule.

166. A city issues bonds for \$500,000.00 on which it pays interest at 5% annually. It accumulates a sinking fund at 4% annually to retire the bonds at the end of fifteen years. Find the annual payment into the

sinking fund and the amount in the fund at the end of five years, without schedule.

167. What sum must be put semiannually into an investment which pays 4% effective in order to accumulate $1,000.00 at the end of five years?

168. A person invests $5,000.00 at 5% annually. Principal and interest are to provide a fixed income for ten years, at the end of which time the capital will be exhausted. The first payment is to be made one year from date. What is the annuity? Set up a schedule.

169. A mortgage for $5,000.00 was given on January 1, 1906, to be repaid in ten equal semiannual instalments with interest, beginning July 1, 1906. Find the periodic payment and set up a schedule, basing on 6% nominal, convertible semiannually.

170. According to the conditions of a will the sum of $20,000.00 is to be held in trust until it has increased to $25,000.00. If the fund can be invested at 5% nominal, compounding semiannually, when will the beneficiary receive the money. Solve without schedule.

171. A father bequeaths his son on his tenth birthday $10,000.00 worth of preferred stock which pays 6% dividends semi-annually. The will further directs that the income shall be invested by the trustee until the son is twenty-one years of age. Assuming that the stock continues to pay dividends regularly, and that these dividends can be invested at 5% nominal, convertible quarterly, find the value of the property at the end of the trustee's term.

172. How large an endowment is necessary for a room in a hospital, costing $5,000.00, to instal and $1,500.00 annually for maintenance? Assume 4% annually.

173. Is it more profitable for a city to pay $3.00 per square yard for paving that will last 5 years or $4.00 for paving that will last 7 years? Assume 5% annually.

174. A business man wishes to set aside on his forty-fifth birthday a sum which will give him an income of $1,000.00 a year for ten years, the first payment to be made on his sixtieth birthday. On a 3½% annual basis, what is the sum necessary?

175. An issue of $50,000.00 of bonds of par value $1,000.00 bearing interest at 5% annually, is to be retired as follows: For the first five years interest only will be paid, after which time the bonds and interest will be amortized over a period of five years. The money to provide for all this is to be raised by equal payments into a fund through the full ten years; the fund can be accumulated at 4% annually. What is the amount of this annual payment? Construct a schedule.

176. A bond redeemable at par $1,000.00 on January 1, 1920, bearing 6% JJ, is bought January 1, 1914, to yield 4%. Find the price?

177. A 6% bond, JJ, redeemable at par $1,000.00 on January 1, 1919, is purchased July 1, 1916, to yield 10%. Find the price.

178. A manufacturing concern contracts for a factory site for $20,000.00 cash and $5,000.00 at the end of each year for five years. If money is worth 5% annually, what would be a fair cash price for the lease?

179. A man borrows $5,000.00, agreeing to pay 6% interest annually, and repay the principal at the end of ten years. He accumulates a sinking fund at 4% annually. Find the annual cost of carrying the debt.

180. Would it be better to amortize the above debt over the ten years by semiannual payments at 5% nominal, semiannually?

181. On his fortieth birthday a man begins to invest \$1,000.00 a year. On his seventieth birthday he makes the usual payment and retires from business.
(a) If money has been worth 4% annually throughout the period, what is the value of his investment?
(b) Assuming that he will live not more than fifteen years, he plans to exhaust the investment in that time. If money will continue to earn 4%, what sum can he withdraw annually?
(c) If he desired to have \$10,000.00 left at the end of the time, what would be his annual allowance?

182. 300 members of the graduating class of a college plan to present their college with a scholarship fund of principal \$100,000.00 on the twentieth anniversary of their graduation. They propose to accumulate this fund by annual payments, the first payment to be made one year after graduation. If the fund can be accumulated at 4% annually, what is the annual payment necessary from each member?

183. What would be the annual payment in the above problem if the first payment were made on the date of graduation, and the last on the twentieth anniversary?

184. Explain the meaning of the terms cable, demand, and 60 day, used in quoting sterling exchange. If the cable rate is 4.20, and interest is reckoned at 4%, exact time, state the demand and 60 day rates. Allow 15 days for transit.

185. A debt of \$200,000.00 is to be paid at the end of twenty-five years by means of a sinking fund into which annual payments are made. If the first pay-

ment is made one year from the date when the debt is incurred, and all sums are invested at an average rate of 4¼% annually, what is the amount of the annual payment?

186. How much can a man afford to spend on a piece of equipment which must be replaced every five years at a cost of $1,000.00 in order to get better equipment which will last eight years? Assume 4% annually.

187. A plant consists of:

X cost	$20,000.00 scrap value	$2,000.00 life	18 years
Y	11,000.00	1,500.00	15
Z	30,000.00	none	5

Find the annual depreciation charge on a 4% annual basis, sinking fund method.

188. Find the composite life of the above plant.

189. A bank loans a farmer $3,000.00 to be repaid with interest at 5½% nominal, payments and interest semiannual for fifteen years. Find the semiannual payment.

190. A man was killed in an accident, and an industrial commission awarded his wife compensation to the amount of $5,000.00, but suggested that the amount be paid at the rate of $50.00 a month. On a 3% effective basis, how long will the payments continue?

191. A man begins at the age of twenty-five to save $10.00 a month and invest his savings in a bank which pays 5% nominal, compounding quarterly. How much will he have when he is sixty years old?

192. If the beneficiary of a life insurance policy for $5,000.00 chooses to accept settlement in twenty-five annual payments, the first to be made at once, what is the annual payment at 3% annually?

193. A 5% JJ bond redeemable at par $1,000.00 twenty-eight years from date is advertised for sale at 92 "to yield about 5.60". How nearly correct is this statement? Is the yield above or below 5.60?

194. A banker wished to remit to his correspondent in Florence, Italy, 55,000 lire. Direct exchange was at 7.05. He bought $7,500 in pesetas, exchange at .203, and instructed his Madrid correspondent to change them to lire after deducting ¼% commission. Madrid quoted lire at 67.50. (That is, 100 lire for 67.50 pesetas).

a) Was the remittance sufficient; what was the margin or deficit in lire?

b) If the remittance was insufficient, what should the banker have paid in dollars?

c) Which was more profitable: direct exchange or arbitration?

195. What sum paid at the end of each year for five years will extinguish three debts, $1,000.00 due in three years, $650.00 due in one year, and $475.00 due in four years. Base on 5% annually.

196. Which is more profitable: to rent an ocean-going steamer for $30,000.00 per year for twenty years, payable in advance, or to buy it for $400,000.00, assuming that it will be worth $5,000.00 as scrap at the end of the twenty years? Base on 4% annually.

197. A debt of $8475.00 bearing interest at 4% annually is being repaid in annual instalments of $1,000.00. In how many years will the debt be discharged?

198. A bond of par $1,000.00, 5% JJ, redeemable at 105 on January 1, 1921, is bought on January 1, 1917, to yield 8%. Find the price and set up a schedule of accumulation of discount.

199. How much must be given to endow a motion picture outfit costing \$3,000.00, scrap value \$500.00 at the end of eight years, annual operating expense \$2,000.00, on a 5% annual basis?

200. A 5% JJ bond redeemable at par \$1,000.00 on January 1, 1918, is purchased May 1, 1911, to yield 4%. What is the price?

201. A woman who has funds on deposit in a savings bank which pays 4% and compounds quarterly is considering the purchase of bonds at par, \$10,000.00, which will net 4.15% annually. Assuming that the bond interest will be paid promptly and that she will deposit it at once in the bank mentioned, what is the difference on the return for the first ten years on a deposit of \$10,000.00?

202. Change f 25,000 to United States money. exchange at 5.16⅝.

203. If a mechanical piano for a dance hall costs \$2,200.00, and saves \$100.00 a month in wages of a pianist and repairs to the piano, in what time will the mechanical piano pay for itself on a 6% effective basis.

204. A farm yields an average annual crop of \$5,000.00. The annual expenses are \$485.00 for equipment, \$350.00 for fertilizer, and \$2,775.00 for wages, board and so on. Find a fair cash price for the farm at 6% annually.

205. Change \$1,250.00 to francs, exchange at 6.57⅝.

206. Set up a schedule showing the reserve against a truck costing \$3,500.00, scrap value \$500.00 at the end of five years, by the method of fixed percentage on a diminishing book value.

207. A manufacturing plant has the following assets:

A wearing value	$40,000.00	life 50 years
B	25,000.00	25
C	5,750.00	18
D	6,750.00	20
E	2,400.00	5

What is the annual depreciation charge on a 4% annual basis, sinking fund method?

208. Change £382 14s. 7d. to dollars, exchange at 4.83875.

209. A corporation issues bonds for $200,000.00, par $1,000.00, paying 5% JJ, redeemable at par in ten years. Is it better for the company to pay the semiannual interest and accumulate a sinking fund by semiannual payments improved at 4% nominal, compounding semiannually, to retire the bonds at maturity; or to buy bonds in the open market on each interest date at an average price of 105 throughout the ten years, so that the semiannual charges for redemptions and interest on outstandings shall be as nearly equal as possible.

210. $1,000.00 is invested in a 60 day bill on Paris exchange at 9.50. At maturity the bill is sold at 9.20. If money is worth 6%, banker's time, what is the net profit on the investment.

211. A debt of $10,000.00 is to be amortized by ten equal biennial payments. If money is worth 6% annual, what is the biennial payment?

212. A charitable organization receives a bequest of $5,000.00 with the understanding that it is all to be used within three years. On a 4¾% annual basis, what is the annual income?

213. Find the equated date for these transactions at 6% exact time:

Debits: May 11, \$500.00; May 19, \$2,000.00, 60 days: June 2, 90-day note for \$500.00.
Credits: May 24, \$350.00; June 12, \$850.00.

214. A 5% bond, AO, par \$1,000.00, redeemable October 1, 1920, at par, is purchased October 1, 1915, at \$950.00. What is the rate of income?

215. What is the price of the above bond to yield 6%?

216. Prepare a schedule for the bond of problem 215, changing to a December 31-June 30 basis, and carrying to maturity.

217. A man pays an annual premium of \$33.38 on an endowment insurance policy. Payments are of course in advance. At the end of twenty years the insurance company pays him \$1,000.00. What sum deposited semiannually in a savings bank which pays 4½% nominal, compounding semiannually, would have accumulated to the same amount? Assume payments at the beginning of the period. On this basis how much of each payment was the cost of the endowment as an investment, and how much was cost of insurance and loading?

218. What is the better yield: A 4% bond redeemable at par, bought at 95; or a 5% bond redeemable at 110, bought at 102.

219. What is the value in English money of a 90-day bill for \$1,500.00 when the demand rate is 4.6875, and the discount rate is 3½%?

220. Which is better for an American importer of English goods: a quotation of 4.65 on London; or 5.17½ on Paris and 25.40 in Paris on London? The face of the invoice is £10,000.

TWO EXAMINATION PAPERS

Do any six examples, in any order.: Tables may be used.

1. Find the balance due on the following account at December 1, on a 5% basis, exact time:

   Debits: July 7, \$500.00, 60 days; August 8, \$600.00, 60 days; October 5, \$500.00, 30 days; November 12, \$450.00, 30 days.

   Credits: August 9, 60-day note without interest, \$1,000.00; November 1, cash, \$400.00; November 20, cash, \$200.00.

2. Balance the following account as of October 1, without interest. State the balance both in dollars and sterling, exchange at 4.85¾. State the apparent profit or loss for the month.

   Debits: Sept. 5, £350 @ 4.86½; Sept. 9, £225 8s. 5d. @ 4.86; Sept. 21, £310 6s. @ 4.86¼.

   Credits: Sept. 8, £300 @ 4.86¼; Sept. 16, £405 10s. 5d. @ 4.8615.

3. What is the cost of the draft in payment of an invoice amounting to f 15,000, exchange at 7.45½.

4. Find the amount and present worth of a 10-year annuity due of \$200.00 a year at 3½% annually.

5. A concern issues \$200,000.00 of serial bonds bearing interest at 6% annually. The trust agreement provides that an annual payment of \$20,000.00 shall be made to meet interest charges on outstandings and to retire as many bonds as possible. In what time will the entire series be retired.? Solve without schedule.

6. A corporation plans to accumulate a sinking fund to retire a bond issue of \$250,000.00 maturing at the end of 25 years. If the trustee can keep the fund invested at an average rate of 4% nominal, compounding semi-

annually, what amount must the corporation pay him at the end of each six months?

7. In the preceding problem what amount ought the trustee's books to show at the end of the fifth year? Solve without schedule.

8. A machine costing $2,000.00 has an estimated life of 12 years and scrap value $500.00. The reserve is to be built up by the method of fixed percentage on diminishing book value. Show the percentage.

9. A 20-year bond redeemable at par $1,000.00 on January 1, 1939, bearing interest at 6% JJ, is purchased January 1, 1919, to yield 5%. What is the price?

10. A $1,000.00 bond bearing semiannual coupons at the rate of 5½%, is redeemable at 110 after 18 years. Find the price to yield 4%.

1. On July 1 your account with your Chicago correspondent shows:

   Debits: March 8, $500.00, 60 days; May 6, $500.00, 60 days; June 5, $250.00, 10 days.

   Credits: April 8, $500.00, 60 days; June 24, $300.00, cash.

   Settlements are made semiannually at 6% exact time. State the balance as of July 1.

2. On what date could the above account be settled with a minimum payment of interest; that is, what is the equated date?

3. a) Change $1,200.00 to sterling at 4.6565.

   b) At the same quotation change £255 13s. 7d. to dollars.

4. You are importing olive oil, and have these quotations per hundred litres:

From Florence, 600 lire, exchange at 7.55.

From Marseilles, 425 francs, exchange at 5.70.

Which is cheaper, and how much?

5. Find the amount of a 15-year annuity of $1.00 a year accumulated at 4¾% annually.

6. A machine costing $3,000.00 has a life of four years and scrap value $200.00.

   a) Find the annual depreciation charge by the sinking fund method, at 4% annually.

   b) Find the annual rate of depreciation by the method of fixed rate on diminishing book value.

7. I have $10,000.00 in a Trust Company which pays 4% nominal, compounding semiannually. Shall I draw the money and buy a 6% stock at 125, dividends payable semiannually, and invest the dividends in the same Trust Company. Which is the better investment and how much over a ten-year period.

8. Value a $1,000.00 bond, redeemable after 13 years at par, on a 7% basis. Interest on the bond 6% JJ.

9. Value a 4% AO bond, par $1,000.00, redeemable at 108 after 8 years, to yield 6%.

10. A series of $1,000,000.00 of bonds, par $1,000.00, bearing interest at 5% annually, is redeemable at par $50,000.00 each year from January 1, 1921, to 1940. Value this series as of January 1, 1920, on a 4% annual basis.

CHAPTER XIV

PROBLEMS FROM EXAMINATIONS OF THE AMERICAN INSTITUTE OF ACCOUNTANTS

This chapter is intended to summarize briefly such parts of the book as are most useful to accountants who are preparing for the examinations of the American Institute of Accountants. The student should first read the paragraphs on reversed multiplication and division in chapter I. These shortened processes are valuable time savers in the rush of an examination, and the results obtained by them are sufficiently accurate. The chapter on averaging accounts should be read, as there was such a problem on a recent examination, and even the matter of equated date is sometimes required on various C. P. A. papers.

The first matter of importance which has not been taken up in the body of the book is the averaging of an account with an English correspondent. This problem occurred on the Institute examination in November, 1918.

A dealer in foreign exchange finds from his books that he has had the following transactions in London exchange during a particular month, viz.:

Exchange bought in the local market:

Jan. 1, 30-day bill payable in London £300 @ 4.75.
15, bill due at sight in London, £2,500 @ 4.76.

Exchange sold in the local market:

Jan. 5, bill due in London at sight £1,000 @ 4.77.
20, cable transfer, £2,000 @ 4.78.

Foreign correspondent's draft honored and paid:

Jan. 20, bill at 30 days after sight accepted Dec. 21, £500 @ 4.78.

State how the balance stands on the account at the close of the month, and how much profit or loss has been derived from the transactions. (At Jan. 31 the rate for cable transfers was 4.78). Is the profit or loss so stated final?

The solution in the Journal of Accountancy for February, 1919, gave the following form as the usual one in the United States. The transactions are listed in both currencies, the sterling being used as an inventory. This inventory is valued at the end of the month, and the profit or loss determined from this.

London Correspondent

Debit	Jan. 1,	£ 300	@ 4.75		$ 1,425
	15,	2,500	@ 4.76		11,900
	20,	500	@ 4.78		2,390
	31,	Cr. Exchange a/c			49
		£3,300			$15,764
Credit	Jan. 5,	£1,000	@ 4.77		$4,770
	20,	2,000	@ 4.78		9,560
	31,	300	@ 4.78	bal.	1,434
		£3,300			$15,764

The balance of sterling on hand January 31 was £300; the value of this at 4.78 was $1,434. This item is entered on the credit side. Then the dollar columns are footed and balanced as usual, and there is found to be a profit of $49.00, which is transferred to Exchange account.

The following comment is made in answer to the last question: "The profit is not final. The balance may not realize 4.78. Besides there is the interest on the thirty-day bill, and on the overdraft caused by the credit of January 20 which will not be covered until the sight draft of January 15 reaches London. Also the London correspondent may charge a commission for handling the business."

A similar problem occurred on the Massachusetts C. P. A. examination for 1914. The problem will be given without solution.

A banking concern dealing in foreign exchange has the following transactions on its account with its London correspondent:

Debits:

Sept. 1, remittance, 30-day bill £400 @ 4.86.
10, remittance, sight bill £100 10s. @ 4.87.
15, remittance, demand bill £200 0s. 6d. @ 4.8675

Credits:

Sept. 2, sight draft £300 @ 4.87½.
12, demand draft, £200 12s. 5d. @ 4.87.
20, cable, demand, £100 @ 4.88.

Ascertain the profit or loss on the account for the month of September and state the balance as of Sept. 30, in foreign and domestic currency, the current rate on that date being 4.89.

Compound interest: The simplest method of arriving at the compound interest on a given amount for a given time and rate is to compute the compound amount first and deduct the principal. All such problems are worked on a basis of one unit: one dollar, one pound sterling, one franc, and so on. One unit at 5% for one unit of time, whether a year, six months or other period, earns .05 of itself, and so at the end of the period amounts to 1.05 of itself. This new principal during the next unit of time earns .05 of itself, and at the end of the period amounts to 1.05 of its value at the beginning of the period. The simplest way to arrive at this amount for two periods is therefore to multiply 1.05, the amount at the end of the first period, by 1.05, the ratio of increase during the second period. Further, note that the ratio of increase during any period is 1.05; that is, the amount at the end of any period is 1.05 of the amount at the beginning. To find the amount at the end of the sixth period, for instance, multiply 1.05 together 6 times. Algebraically expressed, this is $(1.05)^6$, which we read 1.05 to the sixth power. In multiplying, it is of course wise to use reversed multiplication. The 1.05 is called the base, and 6 is called the exponent, showing how many times the base must be used as a factor.

Another shortcut: It is not necessary to multiply five times to obtain this result. When we have multiplied $1.05 \times 1.05 \times 1.05$ we have obtained $(1.05)^3$. Now if we multiply this result by itself, we shall be in effect multiplying by $1.05 \times 1.05 \times 1.05$ all over again, except that we multiply once instead of three times. This leads to an important algebraic truth which was necessary in answering a question which occurred on the 1917 examination. The truth referred to is this: since $(1.05)^3 \times (1.05)^3 = (1.05)^6$, in general whenever exponents are added, bases are multiplied.

The problem was this: You are called upon to state what is the annual sinking fund payment necessary to redeem a principal of $1,000,000.00 due 30 years hence, it being assumed that the annual sums set aside are invested at compound interest at 5% annually. State what computation you would make to arrive at the result desired. You need not work out the computation.

The solution of this problem requires the computation of $(1.05)^{30}$. The shortest solution is:

Find $(1.05)^2$

That quantity multiplied by itself gives (1.05)
" " " " " " $(1.05)^8$
" " " " " " $(1.05)^{16}$
" " " " " " $(1.05)^{32}$

This result is two periods more than the required time. Since adding exponents is the same as multiplying bases, evidently, subtracting exponents is the same as dividing bases. So $(1.05)^{32}$ divided by $(1.05)^2$ will give $(1.05)^{30}$.

The finding of the sinking fund payment will be shown later.

To return to the matter of compound amount, the simplest method of working out the amount of one unit for any rate and time is to multiply 1 plus the rate of interest by itself, or to multiply multiples of this quantity together, until the desired total of exponents is obtained. This compound amount is often indicated by the symbol s^n. This quantity is usually given in problems where it must be used in finding a sinking fund payment, or summing an annuity, or in other cases.

Compound interest for any period is, as has been said, amount less principal. In many problems the interest is in question rather than amount, so this fact should be kept in mind.

Present worth: One very common use for compound amount is in finding present worth. If a dollar will amount in five years at 5% to \$1.27628156, then the principal required to accumulate \$1.00 at the end of five years at 5% must be 1 divided by 1.27628156, or \$.78352617. In general, if 1 amounts in n years at rate i to $(1+i)^n$, or s^n, then the present investment which will yield 1 is $\frac{1}{(1+i)^n}$, or $\frac{1}{s^n}$. The rule for finding present worth then is to divide compound amount into 1. The symbol for this quantity is v^n. It is usually given when it is necessary in working out an amortization factor. But if s^n happens to be given, v^n can easily be found by division. Contracted division should be used in this computation.

Compound discount is simply the discount found by compound interest methods. We are familiar with discount on notes, found by simple interest. It is the difference between present value and the par value at some future date. So compound discount is the difference between 1 and the present worth of 1. Expressed as a rule, subtract the present worth from the par value due at some time in the future. This quantity is important in annuity calculations. The compound discount on 1 for 5 years at 5% is 1—.78352617, or .21647383.

Annuities: An annuity is a series of equal payments made at equal intervals of time, improved at a fixed rate of interest. The rule for finding the amount of an annuity is:

> Divide the compound interest for the given rate and time by the interest for one period, and multiply by the periodic payment.

This is demonstrated with schedules at the beginning of chapter VII. For instance, the amount of payments of 1 each at the end of each year for 15 years accumulated at 4% annually, given that the compound amount of 1 for the same time and rate is 1.80094351, is found as follows:

compound amount	1.80094351
compound interest	.80094351
single interest	.04
quotient	20.02358775

Note that the payments are held to be made at the end of the period. This is because in the business world such payments are usually made out of profits, which are reckoned at the end of the fiscal period. If the payments are made at the beginning of the period, the result is an annuity due, as it is called. The amount of such an annuity can be found from the amount of an ordinary annuity by multiplying by 1 + i. In the above annuity, if the payments were made at the beginning of each period, the amount would be 20.02358775 × 1.04 = 20.82453126.

The present worth, or cash value, of an annuity is a term used in two senses. It may mean first the cash payment which will extinguish a debt which was intended to be repaid in annuity form with inteiest charged on balances outstanding. Second it may mean the present investment which will yield a given income; the interest on the portion remaining invested increases that protion and therefore increases the term during which the income can continue. This last is the annuity in the banking and insurance sense. The rule for valuing the present worth of an ordinary annuity is:

> Divide the compound discount for the given time and rate by the interest for one period, and multiply by the periodic payment.

This is also demonstrated with schedules in chapter VII. For instance, the present worth of the annuity which we summed just above, is found as follows, given that the present worth of 1 for 15 years at 4% annually is .5552645:

present worth	.5552645
compound discount	.4447355
single interest	.04
quotient	11.1183875

The present worth of an annuity due is found by multiplying the present worth of an ordinary annuity by 1 + i. If the above were an annuity due its present worth would by 11.1183875 × 1.04 = 11.563123.

One more sort of annuity which occurs frequently is the **deferred annuity.** This is an annuity of n payments which does not begin until after m periods; that is, the first payment is to be made at the end of the first period after m periods have expired. For instance, the above 15-year annuity might be deferred 5 years. This means that the first payment is made at the end of the 6th year. There are two methods of finding the present worth of such an annuity.

We have already found that at the beginning of the sixth year, which is the same as the end of the fifth year, the present worth is 11.1183875. Since this amount is not due for five years, it must be discounted to the present time. The proper method of doing this is to multiply by the present worth of 1 for 5 years at 4%. If the problem does not state this we must proceed as in the model problem. The present worth is 1 divided by the compound amount. By multiplication we find this compound amount to be 1.2166529, which, divided into 1, gives as quotient the present worth, .82192711. Now multiply this by 11.1183875, and we have the worth of this 15-year annuity, which will not begin for five years; this value is 9.13850408.

The other method is to subtract the present worth of an annuity for the term of deferment from the present worth of an annuity from the present time to the time of the last payment. For the annuity we are discussing, subtract the present worth of a 5-year annuity from that of a 20-year annuity.

If present worth of a 20-year annuity is	13.59032634
and of a 5-year annuity	4.45182233
subtracting, we have	9.13850401

the present worth of a 15-year annuity deferred 5 years, interest at 4% annually.

The following Institute problem included both amount and present worth of an ordinary annuity:

A owns an annuity of $50.00 per annum, the first payment on which falls due one year hence, and continues for a period of 20 years certain. State

a) the present value of the benefit;

b) the amount which he will have accumulated at the end of the period if he invests each moiety as it becomes due.

Assume interest at 4% payable annually. In this connection it is stated that the value of 1.04^{20} is 2.191123.

a) present worth $= \dfrac{\text{compound discount}}{\text{single interest}}$

compound amount	2.191123
by division	.456387 is present worth
compound discount	.543613
divided by	.04
present worth of an annuity of 1	13.5904
times 50	$679.52

If only the present worth had been required very likely the present worth of 1, namely v^{20}, would have been given. In that case the first division would have been avoided, and the solution would begin with finding the compound discount.

b) amount $= \dfrac{\text{compound interest}}{\text{single interest}}$

compound amount	2.191123
compound interest	1.191123
single interest	.04
quotient	29.778075
times 50	$1,488.90

On the May, 1919, examination occurred this problem: A lease has run five years to run at $1,000.00 a year payable at the end of each year, with an extension for a further five years at $1,200.00 a year. On a 6% basis what sum should be paid now in lieu of the ten year's rent? v^5 at 6% = .7473.

for the first five years:		
present worth	.7473	
compound discount	.2527	
divided by .06	4.211667	
times 1,000		$4,211.67
For the last five years, the value at the beginning of the sixth year is, as above	4.211667	
discount this 5 years, multiplying by v^5	.7473	
product	3.1474	
times 1,200		3,776.85
total		$7,988.52

Comment on the solution: The present worth of the rentals for the first five years is found by the rule for present worth: divide the compound discount by the single interest. The last five years' rentals constitute a five-year annuity deferred five years. The present worth at the beginning of the sixth year is the same per dollar as for the first five years. But this must be discounted from the beginning of the sixth year to the beginning of the first year, multiplying by v^5. If more digits had been given in the value of v^5 the problem might have been solved by the second method. $v^{10} = v^5$ times v^5; $.7473 \times .7473 = .5584$, about.

present worth	.5584
compound discount	.4416
divided by .06	7.36
the present worth of a 10-year annuity.	
subtract present worth of a 5-year annuity	4.2117
giving	3.1483
present worth of the deferred annuity.	

times 1,200 = $3,777.96, which is very inaccurate, owing to the lack of sufficient digits.

Sinking funds: We saw that if 1 per annum for 15 years was improved at 4% annually, the sum at the end of the 15 years was 20.02358775. Per contra, if it is desired to make an annual payment for 15 years such that at the end of the 15 years the sum shall be 1, this annual payment is 1 divided by 20.02358775, which is .0499411. Now this 20.02358775 was obtained by dividing the compound interest by the single interest. Evidently the annual payment which will amount to 1 should be obtained by dividing the single interest by the compound interest. This annual payment which is to amount to a given sum is used most commonly in finding payments into the various reserves, whether for sinking funds or for depreciation.

This problem occurred on a recent examination: In auditing the books of a corporation you find that in order to provide a sum to redeem a mortgage of $100,000.00 falling due at the end of ten years, a reserve of $8,000.00 per annum has been set aside for three years, but that contrary to intention the company has failed to accumulate interest thereon. Assuming interest at 4% convertible annually, what should have been the total accumulations to date and what amount should now be set aside for the next seven years in order to complete the sinking fund. $1.04^7 = 1.31593$.

The answer to the first question requires the summing of a three-year annuity of $8,000.00 at 4%. First we must find 1.04^3; then

compound amount	1.124864
compound interest	.124864
single interest	.04
quotient	3.1216
times 8,000	$24,972.80

This is the amount which ought to be in the fund. Whether it is advisable to assume that the interest of $972.80 is added to the fund by an adjusting entry is questionable. There is another question which depends on the date at which the audit actually takes place. If it took place well along in the next fiscal year, we might assume that the amount in the fund would be increased by one year's interest and brought on the books at the end of the year. The remainder of the solution given here assumes that no interest was added, but that the fund was left at $24,000.00, and that the remaining payments must supply the balance of $76,000.00. The rule for finding a sinking fund payment has been given: divide the single interest by the compound interest:

compound amount	1.31593
compound interest	.31593
dividing into	.04
quotient	.1266

This is the sinking fund payment per dollar. Multiplying by 76,000 we have the annual payment, $9,621.60.

On the Michigan C. P. A. examination in 1915 was this problem:

A contractor proposes to build a bridge to Belle Isle and accepts the city's 4% 20-year bonds in payment to the amount of $2,000,000.00. He advocates as a means of retiring the bonds the establishment of a toll system on foot passengers and automobiles at the respective rates of one and five cents each. Assuming the ratio of foot passengers to automobiles to be ten to one, how many of each would be necessary to pay the interest annually and create a fund which, placed at the same rate of interest, would be sufficient to retire the bonds at maturity. $1.00 compounded at 4% for 20 years will amount to 2.19112314.

The solution will not be given, but the number of foot passengers is 9,810,893.

A different type of sinking fund problem is the following: Argument has been strongly urged that, aside from any question of possible mismanagement or of the difficulty of making satisfactory investments to yield the same rate of interest as the bonds, a sinking fund for bonds is more expensive than an arrangement for serial repayment of the bonds. This is illustrated by the case of $20,000.00 of 5% bonds. If these are paid off in a series, one each year, the total payment will be principal $20,000.00, interest $10,500.00, total $30,500.00.

The annual sinking fund to pay off these bonds would on a 5% annual basis amount to $604.85, making in 20 years $12,097.00, and the interest paid on the bonds would be $20,000.00, total payments $32,097.00. The apparent excess burden is accordingly $1,297.00. Discuss the above argument and show clearly just what the figures mean and in what the apparent saving actually consists.

The solution given in the "Journal of Accountancy" read as follows:

"If the bonds are serial, one bond being paid off each year, the average capital of which the company would have the use would be $10,500.00. As the interest is the same amount, the company paid 100% in 20 years or 5% per annum.

"By the sinking fund method the bonds were virtually paid off at the rate of $604.85 per year. This means that the available capital in use was diminished by that amount at the end of each year—that is, that the company had the use of $20,000.00 the first year, $19,395.15 the second year, and so on. As the last payment was made at the end of the twentieth year, they had the use of $20,000.00 less 19 × $604.85, or $8,507.85. This makes an average capital in use of $14,253.92. For the use of this average capital the company paid interest $20,000.00, against which there is a credit of $7,903.00, the compound interest realized from the sinking fund. This means that the net interest charge was $12,097.00. $12,097.00 for the use of $14,253.92 means a rate of a trifle less than 84.87% for twenty years, or 4.1435% per annum. The advantage of the sinking fund method is apparent, and is explained by the fact that the fund earns compound interest.

"The apparent excess burden mentioned in the problem is not a true excess. It is reached by calling the $20,000.00 paid for coupons all interest, which is not true. Of this amount $7,903.00 was applied to the principal, being the difference between the face of the bonds, $20,000.00, and the actual payments to the sinking fund, $12,097.00. The true amount of interest was therefore $12,097.00, not $20,000.00. The interest paid must be considered in relation to the capital of which the corporation had the use during the 20 years, not in relation to the face of the bonds."

This closes the quotation. The solution does not require any computation, but does bring out the idea of average capital, and is an interesting comparison of redemption by sinking fund and serial redemption in equal amounts. If, however, the redemption had been on an amortization principle, by which the annual total payments of interest and principal together had been as nearly equal as possible, there would be practically no difference in dollars and cents between amortization and the sinking fund redemption.

Depreciation: As has been said, the depreciation reserve is usually built up on the sinking fund principle.

The determination of the periodic payment under those conditions would be by the process just demonstrated.

Another method of reckoning depreciation is by computing a fixed percentage periodically, based on a diminishing balance. After the percentage has been determined, the annual deduction from the net book value as at the beginning of the year can be easily found. This method is sometimes required in accounting problems, but the percentage is fixed arbitrarily, and has no reference to any estimated life of the asset. There is a formula by which the percentage can be computed, but the computation requires the use of logarithms unless the figures in the problem are "doctored". The following problem illustrates this:

A machine costing \$81.00 is estimated to have a life of four years with a residual value of \$16.00. Prepare a statement showing the annual charge for depreciation under each of these methods:

a) straight line;

b) constant percentage of diminishing value;

c) annuity method.

For convenience in arithmetical calculation assume the rate of interest to be 10%.

a) The wearing value, as it is called, is \$65.00; that is, that amount of capital is sunk in the machine and will never be realized. In a life of four years, the annual consumption of capital by the straight-line method is 1/4 of that amount, or \$16.25.

b) The formula for determining the constant percentage is $r = 1 - \sqrt[n]{\frac{S.}{C}}$

Expressed as a rule: Divide the scrap value by the cost; find the number which, multiplied together n times, will give that amount; subtract from 1.

In this problem, S is 16; C is 81; n is 4.

$$\frac{16}{81}=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$$

so the fourth root of $\frac{16}{81}$ is $\frac{2}{3}$, or as a decimal, $.66\frac{2}{3}$, or as a percent, $66\frac{2}{3}\%$.

Subtracting from 1, we have the annual rate, $33\frac{1}{3}\%$.

It is well to subjoin this schedule:

Schedule Showing Depreciation Charged Against Asset......, Costing \$81.00, Life 4 Years, Estimated Scrap Value, \$16.00

Charged at $33\frac{1}{3}\%$ Annually Based on Net Book Value at Beginning of Each Year

Year	Value at Beginning of Year		Depreciation $33\frac{1}{3}\%$	Amount in Reserve at End of Year
1	\$81.00		\$27.00	\$27.00
2	54.00		18.00	45.00
3	36.00		12.00	57.00
4	24.00		8.00	65.00
Remainder	\$16.00	Total	\$65.00	

The computation by the sinking fund method is:

Compound amount, 4 years, at 10% annually	1.4641
compound interest	.4641
divide into	.10
quotient	.21547
times 65	\$14.00555

The schedule, omitting title, is:

Year	Interest on Fund, 10%	Annual Payment	Total Addition to Fund	Amount in Fund
1		$14.00555	$14.00555	$14.00555
2	$1.4006	14.00555	15.40615	29.41170
3	2.9412	14.00555	16.94675	46.35845
4	4.6358	14.00555	18.64135	64.99980
	$8.9776	$56.02220	$64.99980	

The annuity method assumes interest on the net book value. It is developed by the following reasoning: The machine at first cost a certain amount. At the end of the life of the machine the amount of capital that has been tied up is not merely the original cost, but the compound amount of that cost. Of this compound amount a certain part is returned in the form of scrap allowance. The balance is total loss, and should be spread over the life of the machine in the form of a sinking fund charge. Expressed as a rule: Multiply the cost by the compound amount; subtract the scrap value; multiply by the sinking fund factor.

For this machine:

cost	$ 81.00
compound amount factor	1.4641
product	118.5921
less scrap	16.00
loss of capital	102.5921
sinking fund factor	.21547
product	22.1055

The schedule is:

Year	Net Book Value at Beginning	Interest 10%	Annual Depreciation Charge	Net Reduction of Book Value
1	$81.00	$ 8.10	$22.1055	$14.0055
2	66.9945	6.6995	22.1055	15.4060
3	51.5885	5.1588	22.1055	16.9467
4	34.6418	3.4642	22.1055	18.6413
	$16.0005	$23.4225	$88.4220	$64.9995

Amortization: Just as the sinking fund factor is the reciprocal of the amount of an annuity, so the amortization factor is the reciprocal of the present worth of an annuity. We saw that the cash payment which would yield an income of one per annum for 15 years if interest was allowed on balances at the rate of 4% per annum was 11.1183875. If on the other hand we invest 1 now and desire an annual income for 15 years on a 4% annual basis, evidently that annual income is 1 divided by 11.1183875, which is .0899411. The rule for finding the present worth of an annuity was to divide the compound discount by the single interest; so the rule for finding the amortization factor is to divide the single interest by the compound discount. But the finding of compound discount may present difficulties, when the compound amount is given, as it was in the problem on the Institute paper of 1918; because the present worth must be found first, with possibilities for mistakes in the work and probability of error in the last digits; then the compound discount must be found by subtraction. But note that the amortization factor is .0899411, while the sinking fund factor is .0499411, a difference of .04, the interest rate. This fact can be proved by algebra, and is investigated in a schedule in chapter X. The simplest way to find the amortization factor, given the compound amount, is to find the sinking fund factor first and add the interest rate. That method will be employed in solving the Institute problem referred to:

A corporation wants to retire a debt of $105,000.00 bearing 5% interest payable annually. The tenth payment, including interest, is to be $15,000.00. The other nine periodical payments are all to include interest and to be of the same amount. Required the amount of each of the nine payments. $1.05^9 = 1.551328$.

The payment of $15,000.00 at the end of the tenth year represented a balance due at the beginning of the year plus 5% interest on that balance accrued during the year. In other words, the $15,000.00 is 105% of balance due at the beginning of the year.

15,000 ÷ 1.05	=	$14,285.71
from 105,000	=	90,714.29

the amount to be amortized over nine years.

To find the amortization factor:

compound amount	1.551328
compound interest	.551328
divided into	.05
sinking fund factor	.0906901
add .05	.1406901
multiply by 90,714.29	$12,762.6025

the annual amortization of principal and interest. To which must be added every year the 5% interest on $14,285.71	714.2855
Total annual cost	$13,476.8880

But if the problem had stated the present worth of 1 for nine years at 5%, .644609, it would have been a simple matter to find the amortization factor directly:

present worth	.644609
compound discount	.355391
divided into .05	.1406901 as above

In general, whenever the amount of 1, or s^n, is given, it is easier to find the amount of an annuity or the sinking fund factor; whenever the present worth of 1, or v^n, is given, it is easier to find the present worth of an annuity or the amortization factor. The four rules are:

$$\text{amount of annuity} \frac{\text{compound interest}}{\text{single interest}}.$$

sinking fund factor $\dfrac{\text{single interest}}{\text{compound interest}}$

present worth of annuity $\dfrac{\text{compound discount}}{\text{single interest}}$

amortization factor $\dfrac{\text{single interest}}{\text{compound discount;}}$ or

amortization factor = sinking fund factor plus single interest.

Valuation of bonds: The handiest formula to use in bond valuation is that of Mr. Makeham. It requires only the finding of v^n, the present worth of 1. It is as follows:

Let C be the redemption value;
c the coupon rate;
n the life of the bond;
i the desired income rate;
K the present worth of C at rate i.

The value of this bond consists of the present worth of the redemption value, which is K, and the present worth of the coupons at the desired yield rate i. If the yield rate were the same as the coupon rate, the bond would obviously be purchased at par. The excess over par, called the premium, or the deficiency under par, called the discount, depend on the relation existing between these rates. If the purchaser is satisfied with a yield rate less than the coupon rate, he will be willing to pay more than par for the bond. If he desires a larger yield than the coupon rate, he will purchase at a discount. The measure of this premium or discount is the ratio of coupon to yield. It should also be kept in mind that most bonds pay semiannual coupons, and that consequently the annual coupon and yield rates must be halved, and the time must be doubled, to conform with the half-year periods. The following problem will illustrate this:

Value a 20-year 5% bond, redeemable at par \$1,000.00, on a 4% basis. Given that v^{40} at 2% is .45289.

C	=	\$1,000.00
c	=	.025
n	=	40
i	=	.02
K = 1,000 × .45289 =		452.89
C — K = 547.11		

If the bond were purchased to yield 5%, that is, to yield the coupon rate, then \$547.11 would be the present worth of the coupons. This can be proved by computing the present worth of a 40-period annuity of \$25.00 at 2½%. But the yield is only 2%, so that the coupons must have cost

$\frac{.025}{.02}$ of \$547.11	=	\$ 683.89
adding the above		452.89
cost of the bond		\$1,136.78

Expressed as a formula, Makeham's rule is,

$$\text{Value of bond} = Cv^n + \frac{c}{i}(C - K).$$

Rule: Find the present worth of the redemption value; call this K. Subtract this from the redemption value, multiply by the coupon rate, and divide by the income rate. Add this last result to K.

Another problem will illustrate this for a bond bought at a discount:

Value a 15-year 5% bond redeemable at \$1,000.00 on a 6% basis. v^{30} at 2½% = .476743.

C	=	\$1,000.00
c	=	.025
n	=	30
i	=	.03

$K = 1{,}000 \times .476743 =$	476.743
$C - K = 523.257$	
$\frac{.025}{.03} \times 523.257 =$	436.048
value	$912.791

Valuation by discounting: Up to the present time the Institute problems have all treated short-term bonds. In such cases the formula is not necessary, it being easier to discount from maturity. For instance, a 5% bond, redeemable at 1,000, is bought on a 4% basis; that is, the semiannual yield is 2%. At maturity the holder will receive par $1,000.00 plus coupon $25.00. So at the beginning of the last half-year the value was 1,025 ÷ 1.02. At that time a coupon was payable to the amount of $25.00. This increased the value of the bond by that amount. Going back to the beginning of the previous half-year, the value is found by discounting, that is, dividing by 1.02; and another coupon again increases the value of the bond. Presented in a vertical schedule

At maturity, par	$1,000.00
plus coupon	25.00
	$1,025.00
value 6 months before, divide by 1.02	1,004.9019
plus coupon	25.00
	$1,029.9019
value 6 months before that date, divide by 1.02	1,009.7078
plus coupon	25.00
	$1,034.7078

Continue until the date stated in the problem has been reached. This method is slightly less accurate than the valuation by formula, because of the error in the right hand digits caused by the division. The method is useful only in cases where speed is necessary, and only when the term of the bond is short.

The above problem occurred on the Institute examination of November, 1918. The redemption value of the bond in that problem was $10,000.00, which gives the value one year before maturity $10,347.0780

value 1½ years before maturity, divide by 1.02	10,144.194
if bought ex interest; otherwise add coupon	250.000
Value	$10,394.194

Redemption at a premium or discount: On the May, 1919 examination of the Institute was this problem:

A bond bearing interest at 5% payable annually and repayable in 5 years with a bonus of 10% is for sale. What price can a purchaser pay who desires to realize 6% on his investment? v^5 at 6% is .7473.

The noteworthy feature of this problem is that the bond is redeemable at a 10% premium; that is, the value of C in Makeham's formula is $1,100.00. This has no effect on the yield, for the problem expressly states that the purchaser desires to realize 6% on his investment. But the coupon rate is 5% on the face of the bond, regardless of the redemption value. And $50.00 on 1,000 is not 5% on the investment as purchased. 5% on 1,000 is only $\frac{10}{11}$ of 5% when adjusted to the redemption value 1,100. And since the purchaser bases his valuation on this redemption value, the coupon rate must also be based on it. We can value this bond by Makeham's formula more quickly than by discounting:

C	=	$1,100.00
c	=	4 6/11%
n	=	5
i	=	6%
K = 1,100 × .7473	=	822.03
C — K = 277.97		
$\frac{4\ 6/11}{6} \times 277.97$	=	210.58
Value		$1,032.61

The solution by discounting is:

Value at maturity	$1,100.00
plus coupon	50.00
	$1,150.00
value one year before, divide by 1.06	1,084.9057
plus coupon	50.00
	$1,134.9057
value two years before, divide by 1.06	1,070.6658
plus coupon	50.00
	$1,120.6658
value three years before, divide by 1.06	1,057.2319
plus coupon	50.00
	$1,107.2319
value four years before, divide by 1.06	1,044.5584
plus coupon	50.00
	$1,094.5584
value five years before, divide by 1.06 if purchased ex interest.	1,032.602

Valuation between interest dates is the usual situation in actual practise. In such cases the value as of the last interest date must be first determined. The change in value up to the next interest date consists of two things: that part of the coupon and of the premium or discount proper to the period. The difference between coupon and premium or discount is the net change in the value of the bond. If the bond is below par this difference causes an increase in value; if the value is above par, this difference causes a decrease in value. The simplest method is to find the difference between coupon and discount or premium and add it to the value of the bond if below par, or subtract it if the bond is above par.

If the bond in the last problem had been purchased three months after the date when its value was \$1,032.602, the value would have been:

Value on last interest date		\$1,032.602
1/4 of 6% of that value	15.4890	
less 1/4 of \$50.00	12.50	2.989
adding, because the value is below redemption value		\$1,035.591

Serial bonds have never been required in Institute examinations. In fact, the valuation of a series is usually a complicated process, because redemptions are usually at a premium which varies as the final date approaches, and the amount redeemed at the various dates is often variable. If a series is regular in every way (that is, if the redemptions are equal in amount, interval, and redemption value), the series may be valued by Makeham's formula. In this case K is the present worth of a series of payments, and the present worth of an annuity must be found and used in place of v^n.

Problem: Value a series of 20 \$1,000.00 bonds, redeemable at par, one each year, beginning one year from date, on a 4% annual basis. The bonds pay 5% annually. $1.04^{20} = 2.1911231$.

We must find the present worth of an annuity:

Compound amount	2.1911231
present worth	.4563869
single interest	.04
divided into .4563869 =	13.5903263

Each redemption is \$1,000.00; the present worth is 1,000 × 13.5903263 =	13,590.3263
C—K = 6,409.6737	
$\frac{5}{4} \times 6{,}409.6737 =$	8,012.0921
Value	\$21,602.4184

This closes the discussion from the investor's standpoint. There remains one phase of the bond problem from the standpoint of the issuer: the writing off of premium or discount at which the bonds were originally sold. Such premium or discount ought to be amortized or accumulated on the annuity principle, but many bookkeepers are not competent to do this work, and the preparation of the schedule is a tiresome task. An alternative method, called the "bonds outstanding method," was required in the following problem:

The A company issues \$100,000.00 bonds, maturing as follows:

\$ 50,000.00	in 1	year
75,000	" 2	"
100,000.00	" 3	"
150,000.00	" 4	"
125,000.00	" 5	"

These bonds were issued at 90. State the amount of discount to be written off annually, using the bonds outstanding method.

The solution in the Journal of Accountancy opened with this quotation from Dickinson's "Accounting Practise and Procedure":

"There are various methods in use for determining the proper interest charge to be made to income account under the varying conditions which arise.

"The first and most correct method, which may be called the effective interest method, consists in charging to income account the effective interest calculated from the known conditions of issue upon the whole amount outstanding during the year.

"The second and more common method, which may be called the equal instalment method, is to ignore altogether the effective interest rate; to charge to income account each year the interest actually accrued, together with a proportionate part, according to the term of issue, of the discount on issue or premium on redemption.

"A third method which may be called the bonds outstanding method, which may be safely adopted where by reason of complication in the terms of issue or redemption it is difficult or impracticable to determine the true interest rate, is to distribute the discount or premium over the period in the proportion that the bonds outstanding for each year bear to the sum of the bonds outstanding for all the years of the currency of the loan.

"When a large accumulated surplus is available the practise is frequently adopted of charging the whole discount on issue to profit and loss account. A great objection to this practise is that thereby the true rate of interest on loans during their currency is entirely lost sight of; current fixed charges are understated; and the discount is charged against surplus arising out of previous operations instead of against income from current operations which should meet it.

"The charge to income account by bonds outstanding method is so close to that given by the effective interest method that for all practical purposes it may safely be adopted."

Table Showing Bonds Outstanding Each Year, and Proportion to Total Outstanding during the Entire Term of the Loan, and Corresponding Reduction of Discount

Year	Par of Bonds Outstanding	Proportion of Total	Reduction of Discount
1	$ 500,000.00	20/69	$14,492.75
2	450,000.00	18/69	13,043.48
3	375,000.00	15/69	10,869.57
4	275,000.00	11/69	7,971.01
5	125,000.00	5/69	3,623.19
Totals	$1,725,000.00	69/69	$50,000.00

Since the bonds were issued at 90 there was a discount of 10% of $500,000.00, or $50,000.00. During the first year the bonds outstanding amounted to $500,000.00; at the end of that year $50,000.00 were redeemed, leaving $450,000.00 outstanding during the second year, and so on. The footing of this Outstanding column is $1,725,000.00. Of this amount $\frac{500,000}{1,725,000}$ or $\frac{20}{69}$ were outstanding during the first year, and so that year must bear $\frac{20}{69}$ of the total discount, $50,000.00, or $14,492.75. The remainder of the schedule needs no explanation. Of course the various columns must prove: the total of the fractions must be $\frac{69}{69}$ and the total of the Reduction of Discount column must be $50,000.00.

APPENDIX

LOGARITHMIC TABLES

Table I—The Compound Amount of One Unit.

Table II—The Present Worth of One Unit.

Table III—The Amount of Periodic Payments of One Unit Each.

Table IV—The Present Worth of Periodic Payments of One Unit Each.

Table V—The Periodic Payment That Will Amount to One. The Sinking Fund.

Table VI—Effective Rate Factors.

Table VII—Ten-Place Logarithms of the Interest Ratios.

CONSTRUCTION OF TABLES

The following brief description of the manner in which the tables in this book have been constructed will be based on the previous sections of the book in which the various functions of s^n were discussed. There is nothing difficult or interesting in the construction of tables, and in fact only their usefulness can justify the manual labor required in their construction.

It has seemed desirable to include all the values of the various factors for one to one hundred periods, and to eight places of decimals. If more than a hundred periods are specified in any problem, the proper value of s^n or v^n can be found by multiplying two or more table values; if a value of some annuity factor is required, it can then be derived from this value of s^n or v^n by formula. If more than eight places of decimals are necessary for accuracy, it will be necessary either to build a schedule covering the life of the transaction, and adjust errors, or it will be necessary to find the required factor by the use of twelve place logarithms, as will be explained below.

Table I: The Compound Amount of One Unit.

Great care must be used in constructing this table, as it is the basis of two of the other tables, and any error in this table will be especially magnified in the table of s_n. The table is developed by ordinary multiplication, the decimals running to not less than twelve places, and fourteen are better. Every ten periods at most there should be an independent check, either by use of some machine, or, as in this case, by use of the fifteen-place logs of $1+i$ given in the Sprague-Perrine "Accountancy of Investment", the anti-logs being found by use of the tables of factors in the same book. If any error in the value as given by actual multiplication is found, the value, and some preceding values should be adjusted before proceeding with further multiplication. As an instance, the table of s^n at 5%, beginning at 50, is made up as follows:

50	11.467399785753
	.573369989288
51	12.040769775041
	.602038488752
52	12.642808263793
	.632140413190
53	13.274948676983
	.663747433849
54	13.938696110832
	.696934805542
55	14.635630916374
	.731781545819
56	15.367412462193
	.768370623110
57	16.135783085303
	.806789154265
58	16.942572239568
	.847128611978
59	17.789700851546
	.889485042577
60	18.679185894123

Now the logarithmic value should be very carefully calculated, and if any discrepancy is discovered which affects any figure except the two at the right, a suitable adjustment should be made. Such adjustment may be on a proportional basis; that is, if the error were 16, add 16 to the last value, 14 to the preceding one, and so on proceeding backward as far as desired.

Table II: The Present Worth of One Unit.

This table is formed by multiplication, but is peculiar in that the actual work begins with the value for 100 periods. Of course we might begin with one period, dividing 1 by 1+i, dividing this result by 1+i, and so on. But the multiplication is easier, and gives the same results. The logarithmic value for 100 periods is calculated, and multiplication proceeds until the value for 90 periods is reached. This value is checked by logarithms, and the multiplication is resumed. The table for 5%, beginning at 60 periods is:

60	.053535523746
	.002676776187
59	.056212299933
	.002810614996
58	.059022914929
	.002951145746
57	.061974060675
	.003098703033
56	.065072763708
	.003253638185
55	.068326401893
	.003416320094
54	.071742721987
	.003587136099
53	.075329858086
	.003766492904
52	.079096350990
	.003954817549
51	.083051168539
	.004152558427
50	.087203726966

The logarithmic value for 50 periods must now be calculated, and any necessary adjustment made before proceeding with the multiplication.

Table III: The Amount of Periodic Payments of One Unit Each.

As was shown on page 61, such amounts may be found by addition from values of the compound amount. This table has been so constructed, and checked every ten periods by the formula $s_n = \frac{s^n - 1}{i}$. For instance, the value of s^n at 5% for 50 periods has just been given as 11.467399785753. Deducting 1 and dividing by .05 we have s^n=209.34799571506. To find s^n for 51 periods we must add to this value s^n for 50 periods. The reason for this is evident. In any annuity, the payments are made at the end of the period. A payment therefore does not accumulate interest until the end of the following period. The payment made at the end of the first period will have accumulated, at the end of the fifty-first period, only fifty periods interest. The schedule is built as follows:

50	209.34799571506
	11.467399785753
51	220.815395500813
	12.040769775041
52	232.856165275854
	12.642808263793
53	245.498973539647
	13.274948676983
54	258.773922216630

and so on. At 60 periods the value should be calculated from the value of s^n for 60 periods, and any discrepancy should be adjusted over as many periods as may seem necessary.

Table IV: The Present Worth of Periodic Payments of One Unit Each.

This table is formed by addition from the table of v^n in precisely the same manner as Table III is formed from Table I. The checks are made from the corresponding value of v^n according to the formula $a_n = \frac{1-v^n}{i}$. The value of v^n for 50 periods being .087203726966, we deduct from 1 and divide by .05, and the result is a_n for 50 periods, 18.25592546068. The table proceeds by addition until the value for 60 periods is reached. This value should be checked and adjusted in a mananer similar to that described for the table of s_n.

Table V: The Periodic Payment That Will Amount to One.

The Sinking Fund: This table is formed by dividing the successive values of s_n into 1. There is no method of checking such division except by differencing. This method, as applied to bond schedules, was described on pages 104 and 105. The third order of differences should be sufficient to show whether the schedule is accurate. Any considerable variation, particularly in case a difference should be larger than the preceding one, will indicate an error in the schedule.

The periodic payment that will extinguish a debt of one unit—the amortization factor—is often given in place of the sinking fund factor. But the schedule presented on page 95 should make it clear that the amortization factor can always be found by adding i to the sinking fund factor. It is desirable to give the sinking fund factors because they are more often used in this book, especially in the various problems dealing with the valuation of assets.

Table VI: The Effective Rate Factor.

This quantity was discussed at length on pages 51, 52, and 54. The process of finding the values is logarithmic, and the antilogs may be found from the table of factors, or less accurately from the ten-place logs of $1+i$ at the end of this book.

The effective rate factor for use with annuity tables (see pages 63 to 65) is not given because it is seldom required. If it is necessary to change from a table value at an annual rate to an effective rate, the rule is:

In case of s_n or a_n, multiply the table value by i, and divide by the correct value of j_n.

In case of $\frac{1}{s_n}$ or $\frac{1}{a_n}$ multiply by j_n and divide by i.

The Compound Amount of One Unit.

$$s^n = (1+i)^n$$

n	1%	1¼%	1½%	1¾%	2%
1	1.01	1.0125	1.015	1.0175	1.02
2	1.0201	1.02515625	1.030225	1.03530625	1.0404
3	1.030301	1.03797070	1.04567838	1.05342411	1.061208
4	1.04060401	1.05095434	1.06136355	1.07185903	1.08243216
5	1.05101005	1.06408215	1.07728400	1.09061656	1.10408080
6	1.06152015	1.07738318	1.09344326	1.10970235	1.12616242
7	1.07213535	1.09085047	1.01984491	1.12912215	1.14868567
8	1.08285671	1.10448610	1.12649259	1.14888178	1.17165938
9	1.09368527	1.11829218	1.14338998	1.16898721	1.19509257
10	1.10462213	1.13227083	1.16054083	1.18944449	1.21899442
11	1.11566835	1.14642422	1.17794894	1.21025977	1.24337431
12	1.12682503	1.16075452	1.19561817	1.23143931	1.26824179
13	1.13809328	1.17526395	1.21355244	1.25298950	1.29360663
14	1.14947421	1.18995475	1.23175573	1.27491682	1.31947876
15	1.16096896	1.20482918	1.25023207	1.29722786	1.34586834
16	1.17257864	1.21988955	1.26898555	1.31992935	1.37278571
17	1.18430443	1.23513817	1.28802033	1.34302811	1.40024142
18	1.19614748	1.25057739	1.30734064	1.36653111	1.42824625
19	1.20810895	1.26620961	1.32695075	1.39044540	1.45681117
20	1.22019004	1.28203723	1.34685501	1.41477820	1.48594740
21	1.23239194	1.29806270	1.36705783	1.43953681	1.51566634
22	1.24471586	1.31428848	1.38756370	1.46472871	1.54597967
23	1.25716302	1.33071709	1.40837715	1.49036146	1.57689926
24	1.26973465	1.34735105	1.42950281	1.51644279	1.60843725
25	1.28243200	1.36419294	1.45094535	1.54298054	1.64060599
26	1.29525631	1.38124535	1.47270953	1.56998269	1.67341811
27	1.30820888	1.39851092	1.49480018	1.59745739	1.70688648
28	1.32129097	1.41599230	1.51722218	1.62541290	1.74102421
29	1.33450388	1.43369221	1.53998051	1.65385762	1.77584469
30	1.34784892	1.45161336	1.56308022	1.68280013	1.81136158
31	1.36132740	1.46975853	1.58652642	1.71224913	1.84758882
32	1.37494068	1.48813051	1.61032432	1.74221349	1.88454059
33	1.38869009	1.50673214	1.63447919	1.77270223	1.92223140
34	1.40257699	1.52556629	1.65899637	1.80372452	1.96067603
35	1.41660276	1.54463587	1.68388132	1.83528970	1.99988955
36	1.43076878	1.56394382	1.70913954	1.86740727	2.03988734
37	1.44507647	1.58349312	1.73477663	1.90008689	2.08068509
38	1.45952724	1.60328678	1.76079828	1.93333841	2.12229879
39	1.47412251	1.62332787	1.78721025	1.96717184	2.16474477
40	1.48886373	1.64361946	1.81401841	2.00159734	2.20803966
41	1.50375237	1.66416471	1.84122868	2.03662530	2.25220046
42	1.51878989	1.68496677	1.86884712	2.07226624	2.29724447
43	1.53397779	1.70602885	1.89687982	2.10853090	2.34318936
44	1.54931757	1.72735421	1.92533302	2.14543019	2.39005314
45	1.56481075	1.74894614	1.95421301	2.18297522	2.43785421

n	1%	1¼%	1½%	1¾%	2%
46	1.58045885	1.77080797	1.98352621	2.22117728	2.48661129
47	1.59626344	1.79294306	2.01327910	2.26004789	2.53634351
48	1.61222608	1.81535485	2.04347829	2.29959872	2.58707039
49	1.62834834	1.83804679	2.07413046	2.33984170	2.63881179
50	1.64463182	1.86102237	2.10524242	2.38078893	2.69158803
51	1.66017814	1.88428515	2.13682106	2.42245274	2.74541979
52	1.67768892	1.90783872	2.16887337	2.46484566	2.80032819
53	1.69446581	1.93168670	2.20140647	2.50798046	2.85633475
54	1.71141047	1.95583279	2.23442757	2.55187012	2.91346144
55	1.72852457	1.98028070	2.26794398	2.59652785	2.97173067
56	1.74580982	2.00503420	2.30196314	2.64196708	3.03116529
57	1.76326792	2.03009713	2.33649259	2.68820151	3.09178859
58	1.78090060	2.05547335	2.37153998	2.73524503	3.15362436
59	1.79870960	2.08116676	2.40711308	2.78311182	3.21669685
60	1.81669670	2.10718135	2.44321978	2.83181628	3.28103079
61	1.83486367	2.13352111	2.47986807	2.88137306	3.34665140
62	1.85321230	2.16019013	2.51706609	2.93179709	3.41358443
63	1.87174443	2.18719250	2.55482208	2.98310354	3.48185612
64	1.89046187	2.21453241	2.59314442	3.03530785	3.55149324
65	1.90936649	2.24221407	2.63204158	3.08842574	3.62252311
66	1.92846015	2.27024174	2.67152221	3.14247319	3.69497357
67	1.94774475	2.29861976	2.71159504	3.19746647	3.76887304
68	1.96722220	2.32735251	2.75226896	3.25342213	3.84425050
69	1.98689442	2.35644442	2.79355300	3.31035702	3.92113551
70	2.00676337	2.38589997	2.83545629	3.36828827	3.99955822
71	2.02683100	2.41572372	2.87798814	3.42723331	4.07954939
72	2.04709931	2.44592027	2.92115796	3.48720990	4.16114037
73	2.06757031	2.47649427	2.96497533	3.54823607	4.24436318
74	2.08824601	2.50745045	3.00944996	3.61033020	4.32925045
75	2.10912849	2.53879358	3.05459171	3.67351098	4.41583556
76	2.13021975	2.57052845	3.10041058	3.73779742	4.50415126
77	2.15152195	2.60266011	3.14691674	3.80320888	4.59423521
78	2.17303717	2.63519336	3.19412049	3.86976503	4.68611991
79	2.19476754	2.66813328	3.24203230	3.93748592	4.77984231
80	2.21671522	2.70148494	3.29066279	4.00639192	4.87543916
81	2.23888237	2.73525350	3.34002273	4.07650378	4.97294794
82	2.26127119	2.76944417	3.39012307	4.14784260	5.07240690
83	2.28388390	2.80406222	3.44097492	4.22042984	5.17385504
84	2.30672274	2.83911300	3.49258954	4.29428737	5.27733214
85	2.32978997	2.87460191	3.54497838	4.36943740	5.38287878
86	2.35308787	2.91053444	3.59815306	4.44590255	5.49053636
87	2.37661875	2.94691612	3.65212535	4.52370584	5.60034708
88	2.40038494	2.98375257	3.70690723	4.60287070	5.71235402
89	2.42438879	3.02104948	3.76251084	4.68342093	5.82660110
90	2.44863267	3.05881260	3.81894851	4.76538080	5.94313313
91	2.47311900	3.09704775	3.87623273	4.84877496	6.06199579
92	2.49785019	3.13576085	3.93437622	4.93362853	6.18323570
93	2.52282869	3.17495786	3.99339187	5.01996703	6.30690042
94	2.54805698	3.21464483	4.05329275	5.10781645	6.43303843
95	2.57353755	3.25482789	4.11409214	5.19720324	6.56169920
96	2.59927293	3.29551324	4.17580352	5.28815429	6.69293318
97	2.62526565	3.33670716	4.23844057	5.38069699	6.82679184
98	2.65151831	3.37841600	4.30201718	5.47485919	6.96332768
99	2.67803349	3.42064620	4.36654744	5.57066923	7.10259423
100	2.70481383	3.46340427	4.43204565	5.66815594	7.24464612

n	2¼ %	2½ %	2¾ %	3 %	3½ %
1	1.0225	1.025	1.0275	1.03	1.035
2	1.04550625	1.050625	1.05575625	1.0609	1.071225
3	1.06093014	1.07689063	1.08478955	1.092727	1.10871788
4	1.09308332	1.01381289	1.11462126	1.12550881	1.14752300
5	1.11767769	1.13140821	1.14527334	1.15927407	1.18768631
6	1.14282544	1.15969342	1.17676836	1.19405230	1.22925533
7	1.16853901	1.18868575	1.20912949	1.22987387	1.27227926
8	1.19483114	1.21840290	1.24238055	1.26677008	1.31680904
9	1.22171484	1.24886297	1.27654602	1.30477318	1.36289735
10	1.24920343	1.28008454	1.31165103	1.34391638	1.41059876
11	1.27731050	1.31208666	1.34772144	1.38423387	1.45996972
12	1.30604999	1.34488882	1.38478378	1.42576089	1.51006866
13	1.33543611	1.37851104	1.42286533	1.46853371	1.56395606
14	1.36548343	1.41297382	1.46199413	1.51258972	1.61869452
15	1.39620680	1.44829817	1.50219896	1.55796742	1.67534883
16	1.42762146	1.48450562	1.54350944	1.60470644	1.73398604
17	1.45974294	1.52161826	1.58595595	1.65284763	1.79467555
18	1.49258716	1.55965872	1.62956973	1.70243306	1.85748120
19	1.52617037	1.59865019	1.67438290	1.75350605	1.92250312
20	1.56050920	1.63861644	1.72042843	1.80611123	1.98978887
21	1.59562066	1.67958185	1.76774021	1.86029457	2.05943148
22	1.63152212	1.72157140	1.81635307	1.91610341	2.13151158
23	1.66823137	1.76461068	1.86630278	1.97358651	2.20611148
24	1.70576658	1.80872595	1.91762610	2.03279411	2.28332849
25	1.74414632	1.85394410	1.97036082	2.09377793	2.36324498
26	1.78338962	1.90029270	2.02454575	2.15659127	2.44595856
27	1.82351588	1.94780002	2.08022075	2.22128901	2.53156711
28	1.86454499	1.99649502	2.13742682	2.28792768	2.62017196
29	1.90649725	2.04640739	2.19620606	2.35656551	2.71187798
30	1.94939344	2.09756758	2.25660173	2.42726247	2.80679370
31	1.99325479	2.15000677	2.31865828	2.50008035	2.90503148
32	2.03810303	2.20375694	2.38242138	2.57508276	3.00670759
33	2.08396034	2.25885086	2.44793797	2.65233523	3.11194235
34	2.13084945	2.31532213	2.51525626	2.73190530	3.22086033
35	2.17879356	2.37320519	2.58442581	2.81386245	3.33359045
36	2.22781642	2.43253532	2.65549752	2.89827833	3.45026611
37	2.27794229	2.49334870	2.72852370	2.98522668	3.57102543
38	2.32919599	2.55568242	2.80355810	3.07478348	3.69601132
39	2.38160290	2.61957448	2.88065595	3.16702698	3.82537171
40	2.43518897	2.68506384	2.95987399	3.26203779	3.95925972
41	2.48998072	2.75219043	3.04127052	3.35989893	4.09783381
42	2.54600528	2.82099520	3.12490546	3.46069589	4.24125799
43	2.60329040	2.89152008	3.21084036	3.56451677	4.38970202
44	2.44186444	2.86380808	3.29913847	3.67145227	4.54334160
45	2.72175639	3.03790328	3.38986478	3.78159584	4.70235855
46	2.78299590	3.11385086	3.48308606	3.89504372	4.86694110
47	2.84561331	3.19169713	3.57887093	4.01189503	5.03728404
48	2.90963961	3.27148956	3.67728988	4.13225188	5.21358898
49	2.97510650	3.35327679	3.77841535	4.25621944	5.39606459
50	3.04204640	3.43710872	3.88232177	4.38390602	5.58492686

n	2¼%	2½%	2¾%	3%	3½%
51	3.11049244	3.52303644	3.98908562	4.51542320	5.78039930
52	3.18047852	3.61111235	4.09878547	4.65008590	5.98271327
53	3.25203929	3.70139016	4.21150208	4.79041247	6.19210824
54	3.32521017	3.79392491	4.32731838	4.93412485	6.40883202
55	3.40002740	3.88877303	4.44631964	5.08214859	6.63314114
56	3.47652802	3.98599236	4.56859343	5.23461305	6.86530108
57	3.55474990	4.09564217	4.69422975	5.39165144	7.01558662
58	3.63473177	4.18778322	4.82332107	5.55340098	7.35428215
59	3.71651324	4.29247780	4.95596239	5.72000301	7.61168203
60	3.80013479	4.39978975	5.09225136	5.89160310	7.87809090
61	3.88563782	4.50978449	5.23228827	6.06835120	8.15382408
62	3.97306467	4.62252910	5.37617620	6.25040173	8.43920793
63	4.06245862	4.73809233	5.52402105	6.43791379	8.73458020
64	4.15386394	4.85654464	5.67593162	6.63105120	9.04029051
65	4.24732588	4.97795826	5.83201974	6.82998273	9.35670068
66	4.34289071	5.10240721	5.99240029	7.03488222	9.68418520
67	4.44060576	5.22996739	6.15719130	7.24592868	10.02313168
68	4.54051939	5.36071658	6.32651406	7.46330654	10.37394129
69	4.64268107	5.49473449	6.50049319	7.68720574	10.73702924
70	4.74714140	5.63210286	6.67925676	7.91782191	11.11282526
71	4.85395208	5.77290543	6.86293632	8.15535657	11.50177414
72	4.96316560	5.91722806	7.05166706	8.40001727	11.90433624
73	5.07483723	6.06515877	7.24558791	8.65201778	12.32098801
74	5.18902107	6.21678774	7.44484158	8.91157832	12.75222259
75	5.30577405	6.37220743	7.64957472	9.17892567	13.19855038
76	5.42515396	6.53151261	7.85993802	9.45429344	13.66049964
77	5.54721993	6.69480043	8.07608632	9.73792224	14.13861713
78	5.67203237	6.86217044	8.29817869	10.03005991	14.63346873
79	5.79965310	7.03372470	8.52637861	10.33096171	15.14564014
80	5.93014530	7.20956782	8.76085402	10.64089056	15.67573754
81	6.06357357	7.28980701	9.00177751	10.96011727	16.22438835
82	6.20000397	7.57455219	9.24932639	11.28892079	16.79224195
83	6.33950406	7.76391599	9.50368286	11.62758842	17.37997041
84	6.48214920	7.95801389	9.76503414	11.97641607	17.98826938
85	6.62799112	8.15696424	10.03357258	12.33570855	18.61785881
86	6.77712092	8.36088834	10.30949583	12.70577981	19.26948387
87	6.92960614	8.56991055	10.59300696	13.08695320	19.94391580
88	7.08552228	8.78415832	10.88431465	13.47956180	20.64195285
89	7.24494653	9.00376227	11.18363331	13.88394865	21.36442120
90	7.40795782	9.22885633	11.49118322	14.30046711	22.11217595
91	7.57463688	9.45957774	11.80719076	14.72948112	22.88610210
92	7.74506621	9.69606718	12.13188851	15.17136556	23.68711568
93	7.91933020	9.93846886	12.46551544	15.62650652	24.51616473
94	8.09751512	10.18693058	12.80831711	16.09530172	25.37423049
95	8.27970921	10.44160385	13.16054584	16.57816077	26.26232856
96	8.46600267	10.70264395	13.52246085	17.07550559	27.18151006
97	8.65648773	10.97021004	13.89432852	17.58777076	28.13286291
98	8.85125871	11.24446530	14.27642255	18.11540388	29.11751311
99	9.05041203	11.52557693	14.66902417	18.65886600	30.13662607
100	9.25404630	11.81371635	15.07242234	19.21863198	31.19140798

n	4%	4½%	5%	5½%	6%
1	1.04	1.045	1.05	1.055	1.06
2	1.0816	1.092025	1.1025	1.113025	1.1236
3	1.124864	1.14116613	1.157625	1.17424138	1.191016
4	1.16985856	1.19251860	1.21550625	1.23882465	1.26247696
5	1.21665290	1.24618194	1.27628156	1.30696001	1.33822558
6	1.26531902	1.30226012	1.34009564	1.37884281	1.41851911
7	1.31593178	1.36086183	1.40710042	1.45467916	1.50363026
8	1.36856905	1.42210061	1.47745544	1.53468651	1.59384807
9	1.42331181	1.48609514	1.55132822	1.61909427	1.68947896
10	1.48024428	1.55296942	1.62889463	1.70814446	1.79084770
11	1.53945406	1.62285305	1.71033936	1.80209240	1.89829856
12	1.60103222	1.69588143	1.79585633	1.90120749	2.01219647
13	1.66507351	1.77219610	1.88564914	2.00577390	2.13292826
14	1.73167645	1.85194491	1.97993160	2.11609146	2.26090396
15	1.80094361	1.93528244	2.07892818	2.23247649	2.39655819
16	1.87298125	2.02237015	2.18287459	2.35526270	2.54035168
17	1.94790050	2.11337681	2.29201832	2.48480215	2.69277279
18	2.02581652	2.20847877	2.40661923	2.62146627	2.85433915
19	2.10684918	2.30786031	2.52695020	2.76564691	3.02559950
20	2.19112314	2.41171402	2.65329771	2.91775749	3.20713547
21	2.27876807	2.52024116	2.78596259	3.07823415	3.39956360
22	2.36991879	2.63365201	2.92526072	3.24753703	3.60353742
23	2.46471554	2.75216635	3.07152376	3.42615157	3.81974966
24	2.56330416	2.87601383	3.22509994	3.61458990	4.04893464
25	2.66583633	2.00543446	3.38635494	3.81339235	4.29187072
26	2.77246978	3.14067901	3.55567269	4.02312893	4.54938296
27	2.88336858	3.28200956	3.73345632	4.24440102	4.82234594
28	2.99870332	3.42969999	3.92012914	4.47784307	5.11168670
29	3.11865145	3.58403649	4.11613560	4.72412444	5.41838790
30	3.24339751	3.74531813	4.32194238	4.98395129	5.74349117
31	3.37313341	3.91385745	4.53803949	5.25806861	6.08810064
32	3.50805875	4.08998104	4.76494147	5.54726238	6.45338668
33	3.64838110	4.27403018	5.00318854	5.85236181	6.84058988
34	3.79431634	4.46636154	5.25334797	6.17424171	7.25102528
35	3.94608899	4.66734781	5.51601537	6.51382501	7.68608679
36	4.10393255	4.87737846	5.79181614	6.87208538	8.14725200
37	4.26808986	5.09686049	6.08140694	7.25005008	8.63608712
38	4.43881345	5.32621921	6.38547729	7.64880283	9.15425235
39	4.61636599	5.56589908	6.70475115	8.06948699	9.70350749
40	4.80102063	5.81636454	7.03998871	8.51330877	10.28571794
41	4.99306145	6.07810094	7.39198815	8.98154076	10.90286101
42	5.19278391	6.35161548	7.76158756	9.47552550	11.55703267
43	5.40049527	6.63743818	8.14966693	9.99667940	12.25045463
44	5.61651508	6.93612290	8.55715028	10.54649677	12.98548191
45	5.84117568	7.24824843	8.98500779	11.12655409	13.76461083
46	6.07482271	7.57441961	9.43425818	11.73851456	14.59048748
47	6.31781562	7.91526849	9.90597109	12.38413287	15.46591673
48	6.57052824	8.27145557	10.40126965	13.06526017	16.39387173
49	6.83334937	8.64367107	10.92133313	13.78384948	17.37750403
50	7.10666335	9.03263627	11.46739979	14.54196120	18.42015429

n	4%	4½%	5%	5½%	6%
51	7.39095068	9.43910490	12.04076978	15.34176907	19.52536353
52	7.68658871	9.86386463	12.64280826	16.18556637	20.69688534
53	7.99405256	10.30773853	13.27494868	17.07577252	21.93869846
54	8.31381435	10.77158677	13.93869611	18.01494001	23.25502037
55	8.64636692	11.25630817	14.63563092	19.00576171	24.65032159
56	8.99222160	11.76284204	15.36741246	20.05107860	26.12934090
57	9.35191046	12.29216993	16.13578309	21.15388793	27.69710134
58	9.72598688	12.84531758	16.94257224	22.31735176	29.35892742
59	10.11502635	13.42335687	17.78970085	23.54480611	31.12046307
60	10.51962741	14.02740793	18.67918589	24.83977045	32.98769085
61	10.94041250	14.65864129	19.61314519	26.20595782	34.96695230
62	11.37802900	15.31828014	20.59380245	27.64728550	37.06496944
63	11.83315016	16.00760275	21.62349257	29.16788620	39.28886761
64	12.30647617	16.72794487	22.70466720	30.77211994	41.64619967
65	12.79873522	17.48070239	23.83990056	32.46458654	44.14497165
66	13.31068463	18.26733400	25.03189559	34.25013880	46.79366994
67	13.84311201	19.08936403	26.28349037	36.13389643	49.60129014
68	14.39683649	19.94838541	27.59766488	38.12126074	52.57736755
69	14.97270995	20.84606276	28.97754813	40.21793008	55.73200960
70	15.57161835	21.78413558	30.42642554	42.42991623	59.07593018
71	16.19448308	22.76442168	31.94774681	44.76356163	62.62048599
72	16.84226241	23.78882066	33.54513415	47.22555751	66.37771515
73	17.51595290	24.85931758	35.22239086	49.82296318	70.36037806
74	18.21659102	25.97798688	36.98351040	52.56322615	74.58200074
75	18.94525466	27.14699629	38.83268592	55.45420359	79.05692079
76	19.70306485	28.36861112	40.77432022	58.50418479	83.80033603
77	20.49118744	29.64519862	42.81303623	61.72191495	88.82835619
78	21.31083494	30.97923256	44.95368804	65.11662027	94.15805757
79	22.16326834	32.37329802	47.20137244	68.69803439	99.80754102
80	23.04979907	33.83009643	49.56144107	72.47642628	105.79599348
81	23.97179103	35.35245077	52.03951312	76.46262973	112.14375309
82	24.93066267	36.94331106	54.64148878	80.66807436	118.87237828
83	25.92788918	38.60576006	57.37356322	85.10481845	126.00472097
84	26.96500475	40.34301926	60.24224138	89.78558347	133.56500423
85	28.04360494	42.15845513	63.25435344	94.72379056	141.57890449
86	29.16534914	44.05558561	66.41707112	99.93359904	150.07363875
87	30.33196311	46.03808696	69.73792467	105.42994698	159.07805708
88	31.54524163	48.10980087	73.22482091	111.22859407	168.62274050
89	32.80705129	50.27474191	76.88606195	117.34616674	178.74010493
90	34.11933334	52.53710530	80.73036505	123.80020591	189.46451123
91	35.48410668	54.90127503	84.76688330	130.60921724	200.82328190
92	36.90347094	57.37183241	89.00522747	137.79272419	212.88232482
93	38.37960978	59.95356487	93.45548884	145.37132402	225.65526431
94	39.91479417	62.65147529	98.12826328	153.36674684	239.19458017
95	41.51138594	65.47079168	103.03467645	161.80191791	253.54625498
96	43.17184138	68.41697730	108.18641027	170.70102340	268.75903027
97	44.89871503	71.49574128	113.59573078	180.08957969	284.88457209
98	46.69466363	74.71304964	119.27551732	189.99450657	301.97764642
99	48.56245018	78.07513687	125.23929319	200.44420443	320.09630520
100	50.50494818	81.58851803	131.50125785	211.46863567	339.30208351

The Present Worth of One Unit.

$$v^n = \frac{1}{(1+i)^n}$$

n	1%	1¼%	1½%	1¾%	2%
1	.99009901	.98765432	.98522168	.98280098	.98029216
2	.98029605	.97546106	.97066175	.96589777	.96116878
3	.97059015	.96341833	.95631699	.94928528	.94232233
4	.96098034	.95152428	.94218423	.93295851	.92384543
5	.95146569	.93977706	.92826033	.91691524	.90573081
6	.94204524	.92817488	.91454219	.90114254	.88797138
7	.93271805	.91671593	.90102679	.88564378	.87056018
8	.92348322	.90539845	.88771112	.87041157	.85349037
9	.91433982	.89422069	.87459224	.85544135	.83675527
10	.90528694	.88318093	.86166723	.84072860	.82034830
11	.89632372	.87227746	.84893323	.82626889	.80426304
12	.88744923	.86150860	.83638742	.81205788	.78849318
13	.87866260	.85087269	.82402702	.79809128	.77303253
14	.86996297	.84036809	.81184928	.78436490	.75787502
15	.86134947	.82999318	.79985151	.77087459	.74301473
16	.85282126	.81974635	.78803104	.75761631	.72844581
17	.84437749	.80962602	.77638526	.74458605	.71416256
18	.83601731	.79963064	.76491159	.73177990	.70015937
19	.82773992	.78975866	.75360747	.71919401	.68643076
20	.81954447	.78000855	.74247042	.70682458	.67297133
21	.81143017	.77037881	.73149795	.69466789	.65977582
22	.80339621	.76086796	.72068763	.68272028	.64683904
23	.79544179	.75147453	.71003708	.67097817	.63415592
24	.78756613	.74219707	.69954392	.65943800	.62172149
25	.77976844	.73303414	.68920583	.64809632	.60953087
26	.77204796	.72398434	.67902052	.63694970	.59757928
27	.76440392	.71504626	.66898574	.62599479	.58586204
28	.75683557	.70621853	.65909925	.61522829	.57437455
29	.74934125	.69749978	.64935887	.60464697	.56311231
30	.74192292	.68888867	.63976243	.59424764	.55207089
31	.73457715	.68038387	.63030781	.58402716	.54124597
32	.72730411	.67198407	.62099292	.57398247	.53063330
33	.72010307	.66368797	.61181568	.56411053	.52022873
34	.71297734	.65549429	.60277407	.55440839	.51002817
35	.70591420	.64740177	.59386608	.54487311	.50002761
36	.69892495	.63940916	.58508974	.53550183	.49022315
37	.69200490	.63151522	.57644309	.52629172	.48061093
38	.68515337	.62371873	.56792423	.51724002	.47118719
39	.67836967	.61601850	.55953126	.50834400	.46194822
40	.67165314	.60841334	.55126232	.49960098	.45289042
41	.66500311	.60090206	.54311559	.49100834	.44401021
42	.65841892	.59348352	.53508925	.48256348	.43530413
43	.65189992	.58615656	.52718153	.47426386	.42676875
44	.64544546	.57892006	.51939067	.46610699	.41840074
45	.63905492	.57177290	.51171494	.45809040	.41019680
46	.63272764	.56471397	.50415265	.45021170	.40215373
47	.62646301	.55774219	.49670212	.44246850	.39426836
48	.62026041	.55085649	.48936170	.43485848	.38653761
49	.61411921	.54405579	.48212975	.42737934	.37895844
50	.60803882	.53733905	.47500470	.42002883	.37152788

n	1 %	1¼ %	1½ %	1¾ %	2 %
51	.60201864	.53070524	.46798491	.41280475	.36424302
52	.59605806	.52415332	.46106887	.40570492	.35710100
53	.59015649	.51768229	.45425505	.39872719	.35009902
54	.58431336	.51129115	.44754192	.39186947	.34323433
55	.57852808	.50497892	.44092800	.38512970	.33650425
56	.57280008	.49874461	.43441182	.37850585	.32990613
57	.56712879	.49258727	.42799194	.37199592	.32343740
58	.56151365	.48650594	.42166694	.36559796	.31709547
59	.55595411	.48049970	.41543541	.35931003	.31087791
60	.55044962	.47456760	.40929597	.35313025	.30478227
61	.54499962	.46870874	.40324726	.34705676	.29880614
62	.53960358	.46292222	.39728794	.34108773	.29294720
63	.53426097	.45720713	.39141669	.33522135	.28720314
64	.52897126	.45156259	.38563221	.32945587	.28157170
65	.52373392	.44598775	.37993321	.32378956	.27605070
66	.51854844	.44048173	.37431843	.31822069	.27063793
67	.51341429	.43504370	.36878663	.31274761	.26533130
68	.50833099	.42967277	.36333658	.30736866	.26012873
69	.50329801	.42436817	.35796708	.30208222	.25502817
70	.49831486	.41912905	.35267692	.29688670	.25002761
71	.49338105	.41395461	.34746495	.29178054	.24512511
72	.48849609	.40884406	.34233000	.28676221	.24031874
73	.48365949	.40379661	.33727093	.28183018	.23560661
74	.47887078	.39881147	.33228663	.27698298	.23098687
75	.47412949	.39388787	.32737599	.27221914	.22645771
76	.46943514	.38902506	.32253793	.26753724	.22201737
77	.46478726	.38422228	.31777136	.26293586	.21766408
78	.46018541	.37947879	.31307523	.25841362	.21339616
79	.45562912	.37479387	.30844850	.25396916	.20921192
80	.45111794	.37016679	.30389015	.24960114	.20510973
81	.44665142	.36559683	.29939916	.24530825	.20108797
82	.44222913	.36108329	.29497454	.24108919	.19714507
83	.43785063	.35662547	.29061531	.23694269	.19327948
84	.43351547	.35222268	.28632050	.23285761	.18948968
85	.42922324	.34787426	.28208917	.22886242	.18577420
86	.42497350	.34357951	.27792036	.22492621	.18213157
87	.42076585	.33933779	.27381316	.22105770	.17856036
88	.41659985	.33514843	.26976666	.21725572	.17505918
89	.41247510	.33101080	.26577996	.21351914	.17162665
90	.40839119	.32692425	.26185218	.20984682	.16826142
91	.40434771	.32288814	.25798245	.20623766	.16496217
92	.40034427	.31890187	.25416990	.20269057	.16172762
93	.39638046	.31496481	.25041369	.19920450	.15855649
94	.39245590	.31107636	.24671300	.19577837	.15544754
95	.38857020	.30723591	.24306699	.19241118	.15239955
96	.38472297	.30344287	.23947487	.18910190	.14941132
97	38091383	.29969666	.23593583	.18584953	.14648169
98	.37714241	.29599670	.23244909	.18265310	.14360950
99	.37340832	.29234242	.22901389	.17951165	.14079363
100	.36971121	.28873326	.22562944	.17642422	.13803297

n	2¼ %	2½ %	2¾ %	3 %	3½ %
1	.97799511	.97560976	.97323601	.97087379	.96618357
2	.95647444	.95181440	.94718833	.94259591	.93351070
3	.93542732	.92859941	.92183779	.91514166	.90194271
4	.91484335	.90595064	.89716573	.88848705	.87144223
5	.89471232	.88385429	.87315400	.86260878	.84197317
6	.87502427	.86229687	.84978491	.83748426	.81350064
7	.85576946	.84126524	.82704128	.81309151	.78599096
8	.83693835	.82074657	.80490635	.78940923	.75941156
9	.81852161	.80072836	.78336385	.76641673	.73373097
10	.80051013	.78119840	.76239791	.74409391	.70891881
11	.78289499	.76214478	.74199310	.72242128	.68494571
12	.76566748	.74355589	.72213440	.70137988	.66178330
13	.74881905	.72542038	.70280720	.68095134	.63940415
14	.73234137	.70772720	.68399728	.66111781	.61778179
15	.71622628	.69046556	.66569078	.64186195	.59689062
16	.70046580	.67362493	.64787424	.62316694	.57670591
17	.68505212	.65719506	.63053454	.60501645	.55720378
18	.66997763	.64116591	.61365892	.58739461	.53836114
19	.65523484	.62552772	.59723496	.57028603	.52015569
20	.64081647	.61027094	.58125057	.55367575	.50256588
21	.62671538	.59538629	.56569398	.53754928	.48557090
22	.61292457	.58086467	.55055375	.52189250	.46915063
23	.59943723	.56669724	.53581874	.50669175	.45328563
24	.58624668	.55287535	.52147809	.49193374	.43795713
25	.57334639	.53939059	.50752126	.47760557	.42314699
26	.56073000	.52623472	.49393796	.46369473	.40883767
27	.54839117	.51339973	.48071821	.45018906	.39501224
28	.53632388	.50087778	.46785227	.43707675	.38165434
29	.52452213	.48866125	.45533068	.42434636	.36874815
30	.51298008	.47674269	.44314421	.41198676	.35627841
31	.50169201	.46511481	.43128391	.39998714	.34423035
32	.49065233	.45377055	.41974103	.38833703	.33258971
33	.47985558	.44270298	.40850708	.37702625	.32134271
34	.46929641	.43190534	.39757380	.36604490	.31047605
35	.45896959	.42137107	.38693314	.35538340	.29997686
36	.44887002	.41109372	.37657727	.34503243	.28983272
37	.43899268	.40106705	.36649856	.33498294	.28003161
38	.42933270	.39128492	.35668959	.32522615	.27056194
39	.41988530	.38174139	.34714316	.31575355	.26141250
40	.41064575	.37243062	.33785222	.30655684	.25257247
41	.40160954	.36334695	.32880995	.29762800	.24403137
42	.39277216	.35448483	.32000968	.28895922	.23577910
43	.38412925	.34583886	.31144495	.28054294	.22780590
44	.37567653	.33740376	.30310944	.27237178	.22010231
45	.36740981	.32917440	.29499702	.26443862	.21265924
46	.35932500	.32114576	.28710172	.25673653	.20546787
47	.35101809	.31331294	.27941773	.24925876	.19851968
48	.34368518	.30567116	.27193940	.24199880	.19180645
49	.33612242	.29821576	.26466122	.23495029	.18532024
50	.32872608	.29094221	.25757783	.22810708	.17905337

n	2¼ %	2½ %	2¾ %	3 %	3½ %
51	.32149250	.28384606	.25068402	.22146318	.17299843
52	.31441810	.27692298	.24397471	.21501280	.16714824
53	.30749936	.27016876	.23744497	.20875029	.16149589
54	.30073287	.26357928	.23109000	.20267019	.15603467
55	.29411528	.25715052	.22490511	.19676717	.15075814
56	.28764330	.25087855	.21888575	.19103609	.14566004
57	.28131374	.24475956	.21302749	.18547193	.14073434
58	.27512347	.23878982	.20732603	.18006984	.13597520
59	.26906940	.23296568	.20177716	.17482508	.13137701
60	.26314856	.22728359	.19637679	.16973309	.12693431
61	.25735801	.22174009	.19112097	.16478941	.12264184
62	.25169487	.21633179	.18600581	.15998972	.11849454
63	.24615635	.21105541	.18102755	.15532982	.11448747
64	.24073971	.20590771	.17618253	.15080565	.11061592
65	.23544226	.20088557	.17146718	.14641325	.10687528
66	.23026138	.19598593	.16687804	.14214879	.10326114
67	.22519450	.19120578	.16241172	.13800853	.09976922
68	.22023912	.18654223	.15806493	.13398887	.09639538
69	.21539278	.18199241	.15383448	.13008628	.09313563
70	.21065309	.17755358	.14971726	.12629736	.08998612
71	.20601769	.17322300	.14571023	.12261880	.08694311
72	.20148429	.16899805	.14181044	.11904737	.08400300
73	.19705065	.16487615	.13801503	.11557998	.08116232
74	19271458	.16085477	.13432119	.11221357	.07841770
75	.18847391	.15693149	.13072622	.10894521	.07576590
76	.18432657	.15310389	.12722747	.10577205	.07320376
77	.18027048	.14936965	.12382235	.10269131	.07072828
78	.17630365	.14572649	.12050837	.09970030	.06833650
79	.17242411	.14217218	.11728039	.09679641	.06602560
80	.16862993	.13870457	.11414412	.09397710	.06379285
81	.16491725	.13532153	.11108917	.09123990	.06163561
82	.16129022	.13202101	.10811598	.08858243	.05955131
83	.15774105	.12880098	.10522237	.08600236	.05753750
84	.15426997	.12565949	.10240620	.08349743	.05559178
85	15087528	.12259463	.09966540	.08106547	.05371187
86	.14755528	.11960452	.09699795	.07870434	.05189553
87	.14430835	.11668733	.09440190	.07641198	.05014060
88	.14113286	.11384130	.09187533	.07418639	.04844503
89	.13802724	.11106468	.08941638	.07202562	.04680679
90	.13498997	.10835579	.08702324	.06992779	.04522395
91	.13201953	.10571296	.08469415	.06789105	.04369464
92	.12911445	.10313460	.08242740	.06591364	.04221704
93	.12627331	.10061912	.08022131	.06399383	.04078941
94	.12349468	.09816500	.07807427	.06212993	03941006
95	.12077719	.09577073	.07598469	.06032032	.03807735
96	11811950	.09343486	.07395104	.05856342	.03678971
97	11552029	.09115596	.07197181	.05685769	.03554562
98	11297828	.08893264	.07004556	.05520614	.03434359
99	.11049221	.08676355	.06817086	.05359383	.03318222
100	.10806084	.08464737	.06634634	.05203284	.03206011

n	4%	4½%	5%	5½%	6%
1	.96153846	.95693780	.95238095	.94786730	.94339623
2	.92455621	.91572995	.90702948	.89845242	.88999644
3	.88899636	.87629660	.86383760	.85161366	.83961928
4	.85480419	.83856134	.82270427	.80721674	.79209366
5	.82192711	.80245105	.78352617	.76513435	.74725817
6	.79031453	.76789574	.74621540	.72524583	.70496054
7	.75991781	.73482846	.71068133	.68743681	.66505711
8	.73069021	.70318513	.67683936	.65159887	.62741237
9	.70258674	.67290443	.64460892	.61762926	.59189846
10	.67556417	.64392768	.61391325	.58543058	.55839478
11	.64958093	.61619874	.58467929	.55491051	.52678753
12	.62459705	.58966386	.55683742	.52598152	.49696936
13	.60057409	.56427164	.53032135	.49856068	.46883902
14	.57747508	.53997286	.50506795	.47256937	.44230096
15	.55526450	.51672044	.48101710	.44793305	.41726506
16	.53390818	.49446932	.45811152	.42458019	.39364628
17	.51337323	.47317639	.43629669	.40244653	.37136442
18	.49362812	.45280037	.41552065	.38146590	.35034379
19	.47464242	.43330179	.39573396	.36157906	.33051301
20	.45638695	.41464286	.37688948	.34272896	.31180473
21	.43883360	.39678743	.35894236	.32486158	.29415540
22	42195539	.37970089	.34184987	.30792567	.27750510
23	.40572633	.36335013	.32557131	.29187267	.26179726
24	39012147	.34770347	.31006791	.27665656	.24697855
25	.37511680	.33273060	.29530277	.26223370	.23299863
26	.36068923	.31840248	.28124073	.24856275	.21981003
27	.34681657	.30469137	.26784832	.23560450	.20736795
28	.33347747	.29157069	.25509364	.22332181	.19563014
29	.32065141	.27901502	.24294632	.21167944	.18455674
30	.30831867	.26700002	.23137745	.20064402	.17411013
31	.29646026	.25550241	.22035947	.19018390	.16425484
32	.28505794	.24449991	.20986617	.18026910	.15495740
33	.27409417	.23397121	.19987254	.17087119	.14618622
34	.26355209	.22389589	.19035480	.16196321	.13791153
35	.25341547	.21425445	.18129029	.15351963	.13010522
36	.24366872	.20502817	.17265741	.14551624	.12274077
37	23429685	.19619921	.16443563	.13793008	.11579318
38	.22528543	.18775044	.15660536	.13073941	.10923885
39	.21662061	.17966549	.14914797	.12392362	.10305552
40	.20828904	.17192870	.14204568	.11746314	.09722219
41	.20027793	.16452507	.13528160	.11133947	.09171905
42	.19257499	.15744026	.12883962	.10553504	.08652740
43	.18516820	.15066054	.12270440	.10003322	.08162962
44	17804635	.14417276	.11686133	.09481822	.07700908
45	.17119841	.13796437	.11129651	.08987509	.07265007
46	.16461386	.13202332	.10599668	.08518965	.06853781
47	.15828256	.12633810	.10094921	.08074849	.06465831
48	.15219476	.12089771	.09614211	.07653885	.06099840
49	.14634112	.11569158	.09156391	.07254867	.05754566
50	.14071262	.11070965	.08720373	.06876652	.05428836

n	4%	4½%	5%	5½%	6%
51	.13530059	.10594225	.08305117	.06518153	.05121544
52	.13009672	.10138014	.07909635	.06178344	.04831645
53	.12509300	.09701449	.07532986	.05856250	.04558156
54	.12028273	.09283683	.07174272	.05550948	.04300147
55	11565551	.08883907	.06832640	.05261562	.04056742
56	.11120722	.08501347	.06507276	.04987263	.03827115
57	.10693002	.08135260	.06197406	.04727263	.03610486
58	.10281733	.07784938	.05902291	.04480818	.03406119
59	.09886282	.07449701	.05621230	.04247221	.03213320
60	.09506040	.07128901	.05353552	.04025802	.03031434
61	.09140423	.06821915	.05098621	.03815926	.02859843
62	.08788868	.06528148	.04855830	.03616992	.02697965
63	.08450835	.06247032	.04624600	.03428428	.02545250
64	.08125803	.05978021	.04404381	.03249695	.02401179
65	.07813272	.05720594	.04194648	.03080279	.02265264
66	.07512762	.05474253	.03994903	.02919696	.02137041
67	.07223809	.05238519	.03804670	.02767485	.02016077
68	.06945970	.05012937	.03623495	.02623208	.01901959
69	.06678818	.04797069	.03450948	.02486453	.01794301
70	.06421940	.04590497	.03286617	.02356828	.01692737
71	.06174942	.04392820	.03130111	.02333960	.01596921
72	.05937445	.04203655	.02981058	.02117498	.01506530
73	.05709081	.04022637	.02839103	.02007107	.01421254
74	.05489501	.03849413	.02703908	.01902471	.01340806
75	.05278367	.03683649	.02575150	.01803290	.01264911
76	.05075353	.03525023	.02452524	.01709279	.01193313
77	.04880147	.03373228	.02335737	.01620170	.01125767
78	.04692449	.03227969	.02224512	.01535706	.01062044
79	.04511970	.03088965	.02118582	.01455646	.01001928
80	.04338433	.02955948	.02017698	.01379759	.00945215
81	.04171570	.02828658	.01921617	.01307828	.00891713
82	.04011125	.02706850	.01830111	.01239648	.00841238
83	.03856851	.02590287	.01742963	.01175022	.00793621
84	.03708510	.02478744	.01659965	.01113765	.00748699
85	.03565875	.02372003	.01580919	.01055701	.00706320
86	.03428726	.02269860	.01595637	.01000664	.00666340
87	.03296852	.02172115	.01433940	.00948497	.00628622
88	.03170050	.02078579	.01365657	.00899049	.00593039
89	.03048125	.01989070	.01300626	.00852180	.00559472
90	.02930890	.01903417	.01238691	.00807753	.00527803
91	.02818163	.01821451	.01179706	.00765643	.00497928
92	.02709772	.01743016	.01123530	.00725728	.00469743
93	.02605550	.01667958	.01070028	.00687894	.00443154
94	.02505337	.01596132	.01019074	.00652032	.00418070
95	.02408978	.01527399	.00970547	.00618040	.00394405
96	.02316325	.01461626	.00924331	.00585820	.00372081
97	.02227235	.01398685	.00880315	.00555280	.00351019
98	.02141572	.01338454	.00838395	.00526331	.00331150
99	.02059204	.01280817	.00798471	.00498892	.00312406
100	.01980004	.01225663	.00760449	.00472883	.00294723

The Amount of Periodic Payments of One Unit Each.

$$S_n = \frac{s^n - 1}{i}$$

n	1%	1¼%	1½%	1¾%	2%
1	1.00	1.00	1.00	1.00	1.00
2	2.01	2.0125	2.015	2.0175	2.02
3	3.0301	3.03765625	3.045225	3.05280625	3.0604
4	4.060401	4.07562695	4.09090338	4.10623036	4.121608
5	5.10100501	5.12657229	5.15226693	5.17808939	5.20404016
6	6.15201506	6.19065444	6.22955093	6.26870596	6.30812096
7	7.21353521	7.26803762	7.32299419	7.37840831	7.43428338
8	8.28567056	8.35888809	8.43283911	8.50753045	8.58296905
9	9.36852727	9.46337420	9.55933169	9.65641224	9.75462843
10	10.46221524	10.58166637	10.70272167	10.82539945	10.94972100
11	11.56684367	11.71393720	11.86326249	12.01484394	12.16871542
12	12.68250301	12.86036142	13.04121143	13.22510371	13.41208973
13	13.80932804	14.02111594	14.23682960	14.45654303	14.68033152
14	14.94742132	15.19637989	15.45038205	15.70953253	15.97393815
15	16.09689554	16.38633463	16.68213778	16.98444935	17.29341692
16	17.25786449	17.59116382	17.93236984	18.28167721	18.63928525
17	18.43044314	18.81105336	19.20135539	19.60160656	20.01207096
18	19.61474757	20.04611953	20.48937572	20.94463468	21.41231238
19	20.81089504	21.29676893	21.79671636	22.31116579	22.84055863
20	22.09100399	22.56297854	23.12366710	23.70161119	24.29736980
21	23.23919403	23.84501577	24.47052211	25.11638938	25.78331719
22	24.47158598	25.14307847	25.83757994	26.55592620	27.29898354
23	25.71630183	26.45736695	27.22514364	28.02065490	28.84496321
24	26.97346485	27.78808403	28.63352088	29.51101637	30.42186247
25	28.24319950	29.13543508	30.06302361	31.02735915	32.03029972
26	29.52563150	30.49962802	31.51396897	32.57043969	33.67090572
27	30.82087781	31.88087337	32.98667850	34.14042238	35.34432383
28	32.12909669	33.27938428	34.48147868	35.73787977	37.05121031
29	33.45038766	34.69537659	35.99870086	37.36329266	38.79223451
30	34.78489153	36.12906880	37.53868137	39.01715029	40.56897921
31	36.13274045	37.58068216	39.10176159	40.69995042	42.37944079
32	37.49406785	39.05044068	40.68828801	42.41219955	44.22702961
33	38.86900853	40.53857120	42.29861233	44.15441305	46.11157020
34	40.25769863	42.04530334	43.93309152	45.92711527	48.03380160
35	41.66027560	43.57086963	45.59208789	47.73083979	49.99447763
36	43.07687836	45.11550550	47.27596921	49.56612949	51.99436719
37	44.50764714	46.67944932	48.98510875	51.43353675	54.03425453
38	45.95272361	48.26294243	50.71988538	53.33362365	56.11493962
39	47.41225085	49.86622921	52.48068366	55.26696206	58.23723841
40	48.88637336	51.48955708	54.26789392	57.23413390	60.40198318
41	50.37523709	53.13317654	56.08191232	59.23573124	62.61002284
42	51.87898946	54.79734125	57.92314100	61.27235654	64.86222330
43	53.39777935	56.48230802	59.79198812	63.34462278	67.15946777
44	54.93175715	58.18833687	61.68886794	65.45315368	69.50265712
45	56.48107472	59.91569108	63.61420096	67.59858386	71.89271027
46	58.04588547	61.66463722	65.56841397	69.78955908	74.33056447
47	59.62634432	63.43544518	67.55194018	72.00273637	76.81717576
48	61.22260777	65.22838824	69.56521928	74.26278425	79.35351927
49	62.83483385	67.04374310	71.60869757	76.56238298	81.94058966
50	64.46318218	68.88178989	73.68282804	78.90222468	84.57940145

n	1%	1¼%	1½%	1¾%	2%
51	66.10781401	70.74281227	75.78807046	81.28301361	87.27098948
52	67.76889215	72.62709742	77.92489152	83.70546635	90.01640927
53	69.44658107	74.53493613	80.09376489	86.17031201	92.81673745
54	71.14104688	76.46662284	82.29517136	88.67829247	95.67307221
55	72.85245735	78.42245562	84.52959893	91.23016259	98.59653365
56	74.58098193	80.40273633	86.79754292	93.82669043	101.55826432
57	76.32679175	82.40777053	89.09950606	96.46865752	104.58942960
58	78.09005966	84.43786766	91.43599865	99.15685902	107.68121819
59	79.87096026	86.49334101	93.80753863	101.89210405	110.83484256
60	81.66966986	88.57450777	96.21465171	104.67521586	114.05153942
61	83.48636656	90.68168911	98.65787149	107.50703215	117.33257021
62	85.32123023	92.81521022	101.13773956	110.38840522	120.67922161
63	87.17444253	94.97540035	103.65480565	113.32020231	124.09280604
64	89.04618695	97.16259285	106.20962774	116.30330585	127.57466216
65	90.93664882	99.37712526	108.80277215	119.33861370	131.12615541
66	92.84601531	101.61933934	111.43481374	122.42703944	134.74867851
67	94.77447546	103.88958108	114.10633594	125.56951263	138.44365208
68	96.72222022	106.18820084	116.81793098	128.76697910	142.21252513
69	98.68944242	108.51555355	119.57019995	132.02040124	146.05677563
70	100.67633684	110.87199776	122.36375295	135.33075826	149.97791114
71	102.68310021	113.25789773	125.19920924	138.69904653	153.97746936
72	104.70993121	115.67362145	128.07719738	142.12627984	158.05701875
73	106.75703053	118.11954172	130.99835534	145.61348974	162.21815913
74	108.82460083	120.59603599	133.96333067	149.16172581	166.46252231
75	110.91284684	123.10348644	136.97278063	152.77205601	170.79177276
76	113.02197531	125.64228002	140.02737234	156.44556699	175.20760822
77	115.15219506	128.21280852	143.12778293	160.18336441	179.71176038
78	117.30371701	130.81546862	146.27469967	163.98657329	184.30599559
79	119.47675418	133.45066199	149.46882017	167.85633832	188.99211550
80	121.67152172	136.11879526	152.71085247	171.79382424	193.77195780
81	123.88823694	138.82028020	156.00151526	175.80021617	198.64739697
82	126.12711931	141.55553371	159.34153799	179.87671995	203.62034490
83	128.38839050	144.32497788	162.73166106	184.02456255	208.69275181
84	130.67227441	147.12904010	166.17263597	188.24499239	213.86660684
85	132.97899715	149.96815310	169.66522552	192.53927976	219.14393898
86	135.30878713	152.84275501	173.21020390	196.90871716	224.52681776
87	137.66187500	155.75328945	176.80835696	201.35461971	230.01735411
88	140.03849375	158.70020556	180.46048231	205.87832555	235.61770119
89	142.43887868	161.68395813	184.16738955	210.48119625	241.33005521
90	144.86326745	164.70500761	187.92990039	215.16461718	247.15665632
91	147.31190014	167.76382021	191.74884889	219.92999798	253.09978945
92	149.78501914	170.86086796	195.62508162	224.77877295	259.16178523
93	152.28286933	173.99662881	199.55945784	229.71240146	265.34502094
94	154.80569802	177.17158667	203.55284971	234.73236850	271.65192135
95	157.35375501	180.38623150	207.60614246	239.84018495	278.08495978
96	159.92729255	183.64105941	211.72023459	245.03738819	284.64665898
97	162.52656548	186.93657265	215.89603811	250.32554250	291.33959216
98	165.15183111	190.27327980	220.13447868	255.70623947	298.16638400
99	167.80334945	193.65169580	224.43649586	261.18109866	305.12971168
100	170.48138294	197.07234200	228.80304330	266.75176789	312.23200591

n	2¼%	2½%	2¾%	3%	3½%
1	1.00	1.00	1.00	1.00	1.00
2	2.0225	2.025	2.0275	2.03	2.035
3	3.06800625	3.075625	3.08325625	3.0909	3.106225
4	4.13703639	4.15251563	4.16804580	4.183627	4.21494288
5	5.23011971	5.25632852	5.28266706	5.30913581	5.36246588
6	6.34779740	6.38773673	6.42794040	6.46840988	6.55015218
7	7.49062284	7.54743015	7.60470876	7.66246218	7.77940751
8	8.65916186	8.73611590	8.81383825	8.89233605	9.05168677
9	9.85399300	9.95451880	10.05621880	10.15910613	10.36849581
10	11.07570784	11.20338177	11.33276482	11.46387931	11.73139316
11	12.32491127	12.48346631	12.64441585	12.80779569	13.14199192
12	13.60222177	13.79555297	13.99213729	14.19202956	14.60196164
13	14.90827176	15.14044179	15.37692107	15.61779045	16.11303030
14	16.24370788	16.51895284	16.79978639	17.08632416	17.67698636
15	17.60919131	17.93192666	18.26178052	18.59891389	19.29568088
16	19.00539811	19.38022483	19.76397948	20.15688130	20.97102971
17	20.43301957	20.86473045	21.30748892	21.76158774	22.70501575
18	21.89276251	22.38634871	22.89344487	23.41443537	24.49969130
19	23.38534966	23.94600743	24.52301460	25.11686844	26.35718050
20	24.91152003	25.54465761	26.19739750	26.87037449	28.27968181
21	26.47202923	27.18327405	27.91782593	28.67648572	30.26947068
22	28.06764989	28.86285590	29.68556615	30.53678030	32.32890215
23	29.69917201	30.58442730	31.50191921	32.45288370	34.46041374
24	31.36740338	32.34903798	33.36822199	34.42647022	36.66652821
25	33.07316996	34.15776393	35.28584810	36.45926432	38.94985669
26	34.81731628	36.01170803	37.25620892	38.55304225	41.31310168
27	36.60070590	37.91200073	39.28075467	40.70963352	43.75906024
28	38.42422178	39.85980075	41.36097542	42.93092252	46.29062734
29	40.28876677	41.85629577	43.49840224	45.21885020	48.91079930
30	42.19526402	43.90270316	45.69460831	47.57541571	51.62267728
31	44.14465746	46.00027074	47.95121003	50.00267818	54.42947098
32	46.13791226	48.15027752	50.26986831	52.50275852	57.33450247
33	48.17601528	50.35403445	52.65228969	55.07784128	60.34121005
34	50.25997563	52.61288531	55.10022765	57.73017652	63.45315240
35	52.39082508	54.92820745	57.61548391	60.46208181	66.47401274
36	54.56961864	57.30141263	60.19990972	63.27594427	70.00760318
37	56.79743506	59.73394794	62.85540724	66.17422259	73.45786930
38	59.07537735	62.22729665	65.58393094	69.15944927	77.02889472
39	61.40457334	64.78297907	68.38748904	72.23422375	80.72490604
40	63.78617624	67.40255354	71.26814499	75.40125973	84.55027776
41	66.22136521	70.08761737	74.22801897	78.66329753	88.50953748
42	68.71134592	72.83980780	77.26928950	82.02319645	92.60737129
43	71.25735121	75.66080300	80.39419496	85.48389234	96.84862928
44	73.86064161	78.55232308	83.60503534	89.04840911	101.23833131
45	76.52250605	81.51613116	86.90417379	92.71986138	105.78167290
46	79.24426243	84.55403443	90.29403857	96.50145723	110.48403145
47	82.02725834	87.66788530	93.77712463	100.39650095	115.35097255
48	84.87287165	90.85958242	97.35599556	104.40839598	120.38825659
49	87.78251126	94.13107199	101.03328545	108.54064785	125.60184557
50	90.76751776	97.48434879	104.81170079	112.79686729	130.99791016

n	2¼ %	2½ %	2¾ %	3 %	3½ %
51	93.79966416	100.92145751	108.69402256	117.18077331	136.58283702
52	96.91015661	104.44449394	112.68310818	121.69619651	142.36323631
53	100.09063513	108.05560629	116.78189365	126.34708240	148.34594958
54	103.34267442	111.75799645	120.99339573	131.13749488	154.53805782
55	106.66788460	115.55092136	125.32071411	136.07161972	160.94688984
56	110.06791200	119.43969439	129.76703375	141.15376831	167.58003099
57	113.54444000	123.42568765	134.33562717	146.38838136	174.44533207
58	117.09918992	127.51132892	139.02985692	151.78003280	181.55091869
59	120.73392169	131.69911214	143.85317799	157.33343379	188.90520085
60	124.45043493	135.99158995	148.80914038	163.05343680	196.51688288
61	128.25056972	140.39137969	153.90139174	168.94503991	204.39497378
62	132.13620753	144.90116419	159.13368002	175.01339110	212.54879786
63	136.10927220	149.52369329	164.50985622	181.26379284	220.98800579
64	140.17173083	154.26178563	170.03387726	187.70170662	229.72258599
65	144.32559477	159.11833027	175.70980889	194.33275782	238.76287650
66	148.57292605	164.09628852	181.54182863	201.16274055	248.11957718
67	152.91581137	169.19869574	187.53422892	208.19762277	257.80376238
68	157.35641712	174.42866313	193.69142022	215.44355145	267.82689406
69	161.89693651	179.78937971	200.01793427	222.90685800	278.20083535
70	166.53961758	185.28411421	206.51842746	230.59406374	288.93786459
71	171.28675898	190.91621706	213.19768422	238.51188565	300.05068985
72	176.14071105	196.68912249	220.06062054	246.66724222	311.55246400
73	181.10387705	202.60635056	227.11228760	255.06725949	323.45680023
74	186.17871429	208.67150932	234.35787551	263.71927727	335.77778824
75	191.36773538	214.88829706	241.80271709	272.63085559	348.53001083
76	196.67350940	221.26050449	249.45229181	281.80978126	361.72856121
77	202.09866337	227.79201710	257.31222983	291.26407469	375.38906085
78	207.64588329	234.48681752	265.38831615	301.00199693	389.52767798
79	213.31791567	241.34898795	273.68649485	311.03205684	404.16114671
80	219.11756877	248.38271265	282.21287345	321.36301855	419.30678685
81	225.04771497	255.59228047	290.97372747	332.00390910	434.98252439
82	231.11128763	262.98208748	299.97550499	342.96402638	451.20691274
83	237.31129160	270.55663967	309.22483137	354.25294717	467.99915469
84	243.65079567	278.32055566	318.72851423	365.88053558	485.37912510
85	250.13293857	286.27856955	328.49354837	377.85695165	503.36739448
86	256.76092969	294.43553379	338.52712095	390.19266020	521.98525329
87	263.53805060	302.79642214	348.83661678	402.89844001	541.25473715
88	270.46765674	311.36633269	359.42962374	415.98539321	561.19865295
89	277.55317902	320.15041900	370.31393839	429.46495500	581.84060581
90	284.79812555	329.15425328	381.49757170	443.34890365	603.20502701
91	292.20608337	338.38310961	392.98875492	457.64937076	625.31720296
92	299.78072025	347.84268735	404.79594568	472.37885189	648.20330506
93	307.52578644	357.53875453	416.92783418	487.55021745	671.89042074
94	315.44511665	367.47722340	429.39334962	503.17672397	696.40658546
95	323.54263177	377.66415398	442.20166674	519.27202569	721.78081595
96	331.82234099	388.10575783	455.36221257	535.85018646	748.04314451
97	340.28834366	398.80840178	468.88467342	552.92569205	775.22465457
98	348.94483139	409.77861182	482.77900194	570.51346282	803.35751748
99	357.79609010	421.02307712	497.05542449	588.62886670	832.47503059
100	366.84650213	432.54865404	511.72444867	607.28773270	862.61165066

n	4%	4½%	5%	5½%	6%
1	1.00	1.00	1.00	1.00	1.00
2	2.04	2.045	2.05	2.055	2.06
3	3.1216	3.137025	3.1525	3.168025	3.1836
4	4.246464	4.27819113	4.310125	4.34226638	4.374616
5	5.41632256	5.47070973	5.52563125	5.58109103	5.63709296
6	6.63297546	6.71689166	6.80191281	6.88805103	6.97531854
7	7.89829448	8.01915179	8.14200845	8.26689384	8.39383765
8	9.21422626	9.38001362	9.54910888	9.72157300	9.89746791
9	10.58279531	10.80211423	11.02656432	11.25625951	11.49131598
10	12.00610712	12.28820937	12.57789254	12.87535379	13.18079494
11	13.48635141	13.84117879	14.20678716	14.58349825	14.97164264
12	15.02580546	15.46403184	15.91712652	16.38559065	16.86994120
13	16.62683768	17.15991327	17.71298284	18.29679814	18.88213767
14	18.29191119	18.93210937	19.59863199	20.29257203	21.01506593
15	20.02358764	20.78405429	21.57856359	22.40866250	23.27596988
16	21.82453114	22.71933673	23.65749177	24.64113999	25.67252808
17	23.69751239	24.74170689	25.84036636	26.99640269	28.21287976
18	25.64541288	26.85508370	28.13238467	29.48120483	30.90565255
19	27.67122940	29.06356246	30.53900391	32.10267110	33.75999170
20	29.77807858	31.37142277	33.06595410	34.86831801	36.78559120
21	31.96920172	33.78313680	35.71925181	37.78607550	39.99272668
22	34.24796979	36.30337795	38.50521440	40.86430965	43.39229028
23	36.61788858	38.93702996	41.43047512	44.11184669	46.99582769
24	39.08260412	41.68919631	44.50199887	47.53799825	50.81557735
25	41.64590829	44.56521015	47.72709882	51.15258816	54.86451200
26	44.31174462	47.57064460	51.11345376	54.96598051	59.15638272
27	47.08421440	50.71132361	54.66912645	58.98910943	63.70576568
28	49.96758298	53.99333319	58.40258277	63.23351045	68.52811162
29	52.96628630	57.42303316	62.32271191	67.71135353	73.63979832
30	56.08493775	61.00706966	66.43884750	72.43547797	79.05818622
31	59.32833526	64.75238779	70.76078988	77.41942926	84.80167739
32	62.70146867	68.66624524	75.29882937	82.67749787	90.88977803
33	66.20952742	72.75622628	80.06377084	88.22476025	97.34316471
34	69.85790851	77.03025646	85.06695938	94.07712206	104.18375459
35	73.65222486	81.49661800	90.32030735	100.25136378	111.43477987
36	77.59831385	86.16396581	95.83632272	106.76518879	119.12086666
37	81.70224640	91.04134427	101.62813886	113.63727417	127.26811866
38	85.97033626	96.13820476	107.70954580	120.88732425	135.90420578
39	90.40914971	101.46442398	114.09502309	128.53612708	145.05845813
40	95.02551570	107.03032306	120.79977424	136.60561407	154.76196562
41	99.82653633	112.84668760	127.83976295	145.11892285	165.04768356
42	104.81959778	118.92478854	135.23175110	154.10046360	175.95054457
43	110.01238169	125.27640402	142.99333866	163.57598910	187.50757724
44	115.41287696	131.91384220	151.14300559	173.57266850	199.75803188
45	121.02939204	138.84996510	159.70015587	184.11916527	212.74351379
46	126.87056772	146.09821353	168.68516366	195.24571936	226.50812462
47	132.94539043	153.67263314	178.11942185	206.98423392	241.09861210
48	139.26320604	161.58790163	188.02539294	219.36836679	256.56452882
49	145.83373429	169.85935720	198.42666259	232.43362696	272.95840055
50	152.66708366	178.50302828	209.34799572	246.21747645	290.33590458

n	4%	4½%	5%	5½%	6%
51	159.77376700	187.53566455	220.81539550	260.75943765	308.75605886
52	167.16471768	196.97476946	232.85616528	276.10120672	328.28142239
53	174.85130639	206.83863408	245.49897354	292.28677309	348.97830773
54	182.84535865	217.14637261	258.77392222	309.36254561	370.91700620
55	191.15917299	227.91795938	272.71261833	327.37748562	394.17202657
56	199.80553991	239.17426756	287.34824924	346.38324733	418.82234816
57	208.79776151	250.93710959	302.71566171	366.43432593	444.95168905
58	218.14967197	263.22927953	318.85144479	387.58821386	472.64879040
59	227.87565885	276.07459711	335.79401703	409.90556562	502.00771782
60	237.99068520	289.49795398	353.58371788	433.45037173	533.12818089
61	248.51031261	303.52536190	372.26290378	458.29014217	566.11587174
62	259.45072511	318.18400319	391.87604897	484.49609999	601.08282405
63	270.82875412	333.50228333	412.46985141	512.14338549	638.14779349
64	282.66190428	349.50988608	434.09334398	541.31127169	677.43666110
65	294.96838045	366.23783096	456.79801118	572.08339164	719.08286076
66	307.76711567	383.71853335	480.63791174	604.54797818	763.22783241
67	321.07780030	401.98586735	505.66980733	638.79811698	810.02150236
68	334.92091231	421.07523138	531.95329770	674.93201341	859.62279250
69	349.31774880	441.02361679	559.55096258	713.05327415	912.20016005
70	364.29045876	461.86967955	588.52851071	753.27120423	967.93216965
71	379.86207711	483.65381513	618.95493625	795.70112046	1027.00809983
72	396.05656019	506.41823681	650.90268306	840.46468208	1089.62858582
73	412.89882260	530.20705747	684.44781721	887.69023960	1156.00630097
74	430.41477550	555.06637505	719.68020807	937.51320278	1226.36667902
75	448.63136652	581.04436193	756.65371848	990.07642893	1300.94867977
76	467.57662118	608.19135822	795.48640440	1045.53063252	1380.00560055
77	487.27968603	636.55996934	836.26072462	1104.03481731	1463.80593658
78	507.77087347	666.20516797	879.07376085	1165.75673226	1552.63429278
79	529.08170841	697.18440053	924.02744889	1230.87335254	1646.79235035
80	551.24497675	729.55769854	971.22882134	1299.57138693	1746.59989137
81	574.29477582	763.38779497	1020.79026240	1372.04781321	1852.39588485
82	598.26656685	798.74024574	1072.82977552	1448.51044293	1964.53963794
83	623.19722952	835.68355680	1127.47126430	1529.17851729	2083.41201622
84	649.12511870	874.28931686	1184.84482752	1614.28333574	2209.41673719
85	676.09012345	914.63233612	1245.08706889	1704.06891921	2342.98174142
86	704.13372839	956.79071924	1308.34142234	1798.79270977	2484.56064590
87	733.29907752	1000.84637685	1374.75849345	1898.72630880	2634.63428466
88	763.63104063	1046.88446381	1444.49641813	2004.15625579	2793.71234174
89	795.17628225	1094.99426468	1517.72123903	2115.38484986	2962.33508224
90	827.98333354	1145.26900659	1594.60730098	2232.73101660	3141.07518718
91	862.10266688	1197.80611188	1675.33766603	2356.53122251	3330.53969841
92	897.58677356	1252.70738692	1760.10454934	2487.14043975	3531.37208032
93	934.49024450	1310.07921933	1849.10977680	2624.93316394	3744.25440513
94	972.86985428	1370.03278420	1942.56526564	2770.30448796	3969.90966944
95	1012.78464845	1432.68425949	2040.69352892	2923.67123480	4209.10424961
96	1054.29603439	1498.15505117	2143.72820537	3085.47315271	4462.65050459
97	1097.46787577	1566.57202847	2251.91461564	3256.17417611	4731.40953486
98	1142.36659090	1638.06776975	2365.51034642	3436.26375579	5016.29410695
99	1189.06125443	1712.78081938	2484.78586374	3626.25826236	5318.27175337
100	1237.62370461	1790.85595626	2610.02515693	3826.70246679	5638.36805857

The Present Worth of Periodic Payments of One Unit Each.

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

n	1%	1¼%	1½%	1¾%	2%
1	.99009901	.98765432	.98522167	.98280098	.98039216
2	1.97039506	1.96311538	1.95588342	1.94869875	1.94156094
3	2.94098521	2.92653371	2.91220042	2.89798403	2.88388327
4	3.90196555	3.87805798	3.85438465	3.83094254	3.80772870
5	4.85343124	4.81783504	4.78264497	4.74785508	4.71345951
6	5.79547647	5.74600992	5.69718717	5.64899762	5.60143089
7	6.72819453	6.66272585	6.59821396	6.53464139	6.47199107
8	7.65167775	7.56812429	7.48592508	7.40505297	7.32548144
9	8.56601758	8.46234498	8.36051732	8.26049432	8.16223671
10	9.47130453	9.34552591	9.22218455	9.10122291	8.98258501
11	10.36762825	10.21780337	10.07111779	9.92719181	9.78684805
12	11.25507747	11.07931197	10.90750521	10.73954969	10.57534122
13	12.13374007	11.93018766	11.73153222	11.53764097	11.34837375
14	13.00370304	12.77055275	12.54338150	12.32200587	12.10624877
15	13.86505252	13.60054592	13.34323301	13.09288046	12.84926350
16	14.71787378	14.42029227	14.13126405	13.85049677	13.57770931
17	15.56225127	15.22991829	14.90764931	14.59508282	14.29187188
18	16.39826286	16.02954893	15.67256089	15.32686272	14.99203125
19	17.22600850	16.81930759	16.42616837	16.04605673	15.67846201
20	18.04555297	17.59931613	17.16863878	16.75288130	16.35143334
21	18.85696313	18.36969495	17.90013673	17.44754919	17.01120916
22	19.66037934	19.13056291	18.62082437	18.13026948	17.65804820
23	20.45582113	19.88203744	19.33086145	18.80124764	18.29220411
24	21.24338726	20.62423451	20.03040537	19.46068565	18.91392560
25	22.02315570	21.35726865	20.71961120	20.10878196	19.52345647
26	22.79520366	22.08125299	21.39863172	20.74573166	20.12103576
27	23.55960759	22.79629925	22.06761946	21.37172644	20.70689780
28	24.31644316	23.50251778	22.72671671	21.98695474	21.28127236
29	25.06578530	24.20001756	23.37607558	22.59160171	21.84438466
30	25.80770822	24.88890623	24.01583801	23.18584934	22.39645555
31	26.54228537	25.56929010	24.64614582	23.76987651	22.93770152
32	27.26958497	26.24129417	25.26713874	24.34385897	23.46833482
33	27.98969255	26.90496215	25.87895442	24.90796951	23.98856355
34	28.70266589	27.56045644	26.48172849	25.46237789	24.49859172
35	29.40858009	28.20785821	27.07559458	26.00725100	24.99861933
36	30.10750504	28.84726737	27.66068431	26.54275283	25.48884248
37	30.79950994	29.47878259	28.23712742	27.06904455	25.96945341
38	31.48466330	30.10250132	28.80505163	27.58628457	26.44064060
39	32.16303297	30.71851982	29.36458288	28.09462857	26.90258883
40	32.83468611	31.32693316	29.91584520	28.59242955	27.35547923
41	33.49968922	31.92783522	30.45896079	29.08523789	27.79948945
42	34.15810814	32.52131874	30.99405004	29.56780136	28.23479358
43	34.81000806	33.10747529	31.52123157	30.04206522	28.66156233
44	35.45545352	33.68639535	32.04062223	30.50817221	29.07996307
45	36.09450844	34.25816825	32.55233718	30.96626261	29.49015988
46	36.72723608	34.82288222	33.05648983	31.41647431	29.89231360
47	37.35369909	35.38062442	33.55319195	31.85894281	30.28658196
48	37.97395949	35.93148096	34.04255365	32.29380129	30.67311957
49	38.58807871	36.47553670	34.52468339	32.72118063	31.05207801
50	39.19611753	37.01287575	34.99968807	33.14120946	31.42360589

n	1%	1¼%	1½%	1¾%	2%
51	39.79813617	37.54358099	35.46767298	33.55401421	31.78784892
52	40.39419423	38.06773431	35.92874185	33.95971913	32.14494992
53	40.98535072	38.58541660	36.38299690	34.35844632	32.49504894
54	41.56866408	39.09670776	36.83053882	34.75031579	32.83828327
55	42.14719216	39.60168667	37.27146681	35.13544550	33.17478752
56	42.71999236	40.10043128	37.70587864	35.51395135	33.50469365
57	43.38712102	40.59301855	38.13387058	35.88594727	33.82813103
58	43.84863468	41.07952449	38.55553751	36.25154523	34.14522655
59	44.40458879	41.56002419	38.97097292	36.61085526	34.45610441
60	44.95503841	42.03459179	39.38026889	36.96398552	34.76088667
61	45.50003803	42.50330054	39.78351615	37.31104228	35.05969282
62	46.03964161	42.96622275	40.18080408	37.65213000	35.35264002
63	46.57390258	43.42342988	40.57222077	37.98735134	35.63984315
64	47.10287385	43.87499247	40.95785298	38.31680723	35.92141486
65	47.62660777	44.32098022	41.33778619	38.64059678	36.19746555
66	48.14515621	44.76146194	41.71210462	38.95881748	36.46810348
67	48.65857050	45.19650562	42.08089125	39.27156509	36.73343478
68	49.16690149	45.62617839	42.44422783	39.57893375	36.99356351
69	49.67019949	46.05054656	42.80219491	39.88101597	37.24859168
70	50.16851435	46.46967561	43.15487183	40.17790267	37.49861929
71	50.66189539	46.88363023	43.50233678	40.46968321	37.74374440
72	51.15039148	47.29247430	43.84466678	40.75644542	37.98406314
73	51.63405097	47.69627091	44.18193771	41.03827560	38.21966975
74	52.11292175	48.09508238	44.51422434	41.31525857	38.45065661
75	52.58705124	48.48897026	44.84160034	41.58747771	38.67711433
76	53.05648638	48.87799532	45.16413826	41.85501495	38.89913169
77	53.52127364	49.26221760	45.48190962	42.11795081	39.11679578
78	53.98145905	49.64169640	45.79498485	42.37636443	39.33019194
79	54.43708817	50.01649027	46.10343335	42.63033359	39.53940386
80	54.88820611	50.38665706	46.40732349	42.87993474	39.74451359
81	55.33485743	50.75225389	46.70672265	43.12524298	39.94560156
82	55.77708666	51.11333717	47.00169720	43.36633217	40.14274662
83	56.21493729	51.46996264	47.29231251	43.60327486	40.33602610
84	56.64845276	51.82218532	47.57863301	43.83614237	40.52551579
85	57.07767600	52.17005958	47.86072218	44.06500479	40.71128999
86	57.50264951	52.51363909	48.13864254	44.28993099	40.89342156
87	57.92441535	52.85297688	48.41245571	44.51097869	41.07198192
88	58.34001520	53.18812531	48.68222237	44.72824441	41.24704110
89	58.75249030	53.51913611	48.94800233	44.94176355	41.41866774
90	59.16088148	53.84606035	49.20985452	45.15161037	41.58692916
91	59.56522919	54.16894850	49.46783696	45.35784803	41.75189133
92	59.96557346	54.48785037	49.72200686	45.56053860	41.91361895
93	60.36195392	54.80281518	49.97242055	45.75974310	42.07217544
94	60.75440982	55.11389154	50.21913355	45.95552147	42.22762299
95	61.14298002	55.42112744	50.46220054	46.14793265	42.38002254
96	61.52770299	55.72457031	50.70167541	46.33703455	42.52943386
97	61.90861682	56.02426698	50.93761124	46.52288408	42.67591555
98	62.28575923	56.32026368	51.17006034	46.70553718	42.81952505
99	62.65916755	56.61260610	51.39907422	46.88504882	42.96031867
100	63.02887877	56.90133936	51.62470367	47.06147304	43.09835164

n	2¼%	2½%	2¾%	3%	3½%
1	.97799551	.97560976	.97323601	.97087379	.96618357
2	1.93446755	1.92742415	1.92042434	1.91346970	1.89969428
3	2.86989687	2.85602356	2.84226213	2.82861135	2.80163698
4	3.78474021	3.76197421	3.73942787	3.71709840	3.67307921
5	4.67945253	4.64582850	4.61258186	4.57970719	4.51505238
6	5.55447680	5.50812536	5.46236678	5.41719144	5.32855302
7	6.41024626	6.34939060	6.28940806	6.23028296	6.11454398
8	7.24718461	7.17013717	7.09431441	7.01969219	6.87395554
9	8.06570622	7.97086553	7.87767826	7.78610892	7.60768651
10	8.86621635	8.75206393	8.64007616	8.53020284	8.31660532
11	9.64911134	9.51420871	9.38206926	9.25262911	9.00155104
12	10.41477882	10.25776460	10.10420366	9.95400400	9.66333433
13	11.16359787	10.98318497	10.80701086	10.63495533	10.30273849
14	11.89593924	11.69091217	11.49100814	11.29607314	10.92052028
15	12.61216551	12.38137773	12.15669892	11.93793509	11.51741089
16	13.31263131	13.05500266	12.80457315	12.56110203	12.09411681
17	13.99768343	13.71219772	13.43510769	13.16611847	12.65132059
18	14.66766106	14.35336363	14.04876661	13.75351308	13.18968173
19	15.32289590	14.97889134	14.64600157	14.32379911	13.70983742
20	15.96371237	15.58916229	15.22725213	14.87747486	14.21240330
21	16.59042775	16.18454857	15.79294611	15.41502414	14.69797420
22	17.20335232	16.76541324	16.34349987	15.93691664	15.16712483
23	17.80278955	17.33211048	16.87931861	16.44360839	15.62041047
24	18.38903624	17.88498583	17.40079670	16.93554212	16.05836760
25	18.96238263	18.42437642	17.90831795	17.41314769	16.48151459
26	19.52311260	18.95061114	18.40225592	17.87684242	16.89035226
27	20.07150376	19.46401087	18.88297413	18.32703147	17.28536451
28	20.60782764	19.96488866	19.35082640	18.76410823	17.66701885
29	21.13234977	20.45354991	19.80615708	19.18845459	18.03576700
30	21.64532985	20.93029259	20.24930130	19.60044135	18.39204541
31	22.14702186	21.39540741	20.68058520	20.00042849	18.73627576
32	22.63767419	21.84917796	21.10032623	20.38876553	19.06886547
33	23.11752977	22.29188094	21.50883332	20.76579177	19.39020818
34	23.58682618	22.72378628	21.90640712	21.13183667	19.70068423
35	24.04579577	23.14515734	22.29334026	21.48722007	20.00066110
36	24.49466579	23.55625107	22.66991753	21.83225249	20.29049381
37	24.93365848	23.95731811	23.03641609	22.16723543	20.57052542
38	25.36299118	24.34860304	23.39310568	22.49246158	20.84108736
39	25.78287646	24.73034443	23.74024884	22.80821513	21.10249987
40	26.19352221	25.10277505	24.07810102	23.11477197	21.35507234
41	26.59513174	25.46612200	24.40691101	23.41239997	21.59910371
42	26.98790390	25.82060683	24.72692069	23.70135920	21.83488281
43	27.37203316	26.16644569	25.03836563	23.98190213	22.06268870
44	27.74770969	26.50384945	35.34147507	24.25427392	22.28279102
45	28.11511950	26.83302386	25.63647209	24.51871254	22.49545026
46	28.47444450	27.15416962	25.92357381	24.77544907	22.70091812
47	28.82586259	27.46748255	26.20299154	25.02470783	22.89943780
48	29.16954777	27.77315371	26.47493094	25.26670664	23.09124425
49	29.50567019	28.07136947	26.73959215	25.50165693	23.27656450
50	29.83439627	28.36231168	26.99716998	25.72976401	23.45561787

n	2¼%	2½%	2¾%	3%	3½%
51	30.15588877	28.64615774	27.24785399	25.95122719	23.62861630
52	30.47030687	28.92308072	27.49182870	26.16623999	23.79576455
53	30.77780623	29.19324948	27.72927368	26.37499028	23.95726044
54	31.07853910	29.45682876	27.96036368	26.57766047	24.11329511
55	31.37265438	29.71397928	28.18526879	26.77442764	24.26405324
56	31.66029768	29.96485784	28.40415454	26.96546373	24.40971327
57	31.94161142	30.20961740	28.61718203	27.15093566	24.55044761
58	32.21673489	30.44840722	28.82450806	27.33100549	24.68642281
59	32.48580429	30.68137290	29.02628522	27.50583058	24.81779981
60	32.74895285	30.90865649	29.22266201	27.67556367	24.94473412
61	33.00631086	31.13039657	29.41378298	27.84035307	25.06737596
62	33.25800573	31.34672836	29.59978879	28.00034279	25.18587050
63	33.50416208	31.55778377	29.78081634	28.15567261	25.30035797
64	33.74490179	31.76369148	29.95699887	28.30647826	25.41097389
65	33.98034405	31.96457705	30.12846605	28.45289152	25.51784916
66	34.21060543	32.16056298	30.29534409	28.59504031	25.62111030
67	34.43579993	32.35176876	30.45775581	28.73304884	25.72087952
68	34.65603905	32.53831099	30.61582074	28.86703771	25.81727489
69	34.87143183	32.72030340	30.76965522	28.99712399	25.91041053
70	35.08208492	32.89795698	30.91937247	29.12342135	26.00039664
71	35.28810261	33.07107998	31.06508270	29.24604105	26.08733976
72	35.48958691	33.24007803	31.20689304	29.36508752	26.17134276
73	35.68663756	33.40495417	31.34490816	29.48066750	26.25250508
74	35.87935214	33.56580895	31.47922936	29.59288107	26.33092278
75	36.06782605	33.72274044	31.60995558	29.70182628	26.40668868
76	36.25215262	33.87584433	31.73718304	29.80759833	26.47989244
77	36.43242310	34.02521398	31.86100540	29.91028964	26.55062072
78	36.60872675	34.17094047	31.98151377	30.00998994	26.61895721
79	36.78115085	34.31311265	32.09879685	30.10678635	26.68498281
80	36.94978079	34.45181722	32.21294098	30.20076345	26.74877567
81	37.11470004	34.58718752	32.32403015	30.29200335	26.81041127
82	37.27599026	34.71915976	32.43214613	30.38058577	26.86996258
83	37.43373130	34.84796074	32.53736850	30.46658813	26.92750008
84	37.58800127	34.97362023	32.63977469	30.55008556	26.98309186
85	37.73887655	35.09621486	32.73944009	30.63115103	27.03680373
86	37.88643183	35.21581938	32.83643804	30.70985537	27.08869926
87	38.03074018	35.33250671	32.93083995	30.78626735	27.13883986
88	38.17187304	35.44634891	33.02271528	30.86045374	27.18728489
89	38.30990028	35.55741269	33.11213165	30.93247935	27.23409168
90	38.44489025	35.66576848	33.19915489	31.00240714	27.27931564
91	38.57690978	35.77148144	33.28384905	31.07029820	27.32301028
92	38.70602423	35.87461604	33.36627641	31.13621184	27.36522732
93	38.83229754	35.97523516	33.44649776	31.20020567	27.40601673
94	38.95579221	36.07340016	33.52457203	31.26233560	27.44542680
95	39.07657940	36.16917089	33.60055672	31.32265592	27.48350415
96	39.19468890	36.26260574	33.67450775	31.38121934	27.52029386
97	39.31020920	36.35376170	33.74647957	31.43807703	27.55583948
98	39.42318748	36.44269434	33.81652512	31.49327867	27.59018307
99	39.53367968	36.52945790	33.88469599	31.54687250	27.62336529
100	39.64174052	36.61410526	33.95104232	31.59890534	27.65542540

n	4%	4½%	5%	5½%	6%
1	.96153846	.95693780	.95238095	.94786730	.94339623
2	1.88609467	1.87266775	1.85941043	1.84631971	1.83339267
3	2.77509103	2.74896435	2.72324803	2.69793338	2.67301195
4	3.62989522	3.58752570	3.54595050	3.50515012	3.46510561
5	4.45182233	4.38997674	4.32947667	4.27028448	4.21236379
6	5.24213686	5.15787248	5.07569207	4.99553031	4.91732433
7	6.00205467	5.89270094	5.78637340	5.68296712	5.58238144
8	6.73274487	6.59588607	6.46321276	6.33456599	6.20979381
9	7.43533161	7.26879050	7.10787168	6.95219525	6.80169227
10	8.11089578	7.91271818	7.72173493	7.53762583	7.36008705
11	8.76047671	8.52891692	8.30641422	8.09253633	7.88687458
12	9.38507376	9.11858078	8.86325164	8.61851785	8.38384394
13	9.98564785	9.68285242	9.39357299	9.11707853	8.85268296
14	10.56312293	10.22282528	9.89864094	9.58964790	9.29498393
15	11.11838743	10.73954573	10.37965804	10.03758094	9.71224899
16	11.65229561	11.23401505	10.83776956	10.46216203	10.10589527
17	12.16566885	11.70719143	11.27406625	10.86460856	10.47725969
18	12.65929698	12.15999180	11.68958690	11.24607447	10.82760348
19	13.13393940	12.59329359	12.08532086	11.60765352	11.15811649
20	13.59032634	13.00793645	12.46221034	11.95038248	11.46992122
21	14.02915995	13.40472388	12.82115271	12.27524406	11.76407662
22	14.45111533	13.78442476	13.16300258	12.58316973	12.04158172
23	14.85684167	14.14777489	13.48857388	12.87504239	12.30007898
24	15.24696314	14.49547837	13.79864179	13.15169895	12.55035753
25	15.62207994	14.82820896	14.09394457	13.41393266	12.78335616
26	15.98276918	15.14661145	14.37518530	13.66249541	13.00316619
27	16.32958575	15.45130282	14.64303362	13.89809991	13.21053414
28	16.66306322	15.74287351	14.89812726	14.12142172	13.40616428
29	16.98371463	16.02188853	15.14107358	14.33310116	13.59072102
30	17.29203330	16.28888854	15.37245103	14.53374517	13.76483115
31	17.58849356	16.54439095	15.59281050	14.72392907	13.92908600
32	17.87355150	16.78889086	15.80267667	14.90419817	14.08404339
33	18.14764567	17.02286207	16.00254921	15.07506936	14.23022961
34	18.41119776	17.24675796	16.19290401	15.23703257	14.36814114
35	18.66461323	17,46101240	16.37419429	15.39055220	14.49824636
36	18.90828195	17.66604058	16.54685171	15.53606843	14.62098713
37	19.14257880	17.86223979	16.71128734	15.67399851	14.73678031
38	19.36786423	18.04999023	16.86789271	15.80473793	14.84601916
39	19.58448484	18.22965572	17.01704067	15.92866154	14.94907468
40	19.79277388	18.40158442	17.15908635	16.04612469	15.04629687
41	19.99305181	18.56610949	17.29436796	16.15746416	15.13801592
42	20.18562674	18.72354975	17.42320758	16.26299920	15.22454332
43	20.37079494	18.87421029	17.54591198	16.36303242	15.30617294
44	20.54884129	19.01838305	17.66277331	16.45785063	15.38318202
45	20.72003970	19.15634742	17.77406982	16.54772572	15.45583209
46	20.88465356	19.28837074	17.88006650	16.63291537	15.52436990
47	21.04293612	19.41470884	17.98101571	16.71366386	15.58902821
48	21.19513088	19.53560654	18.07715782	16.79020271	15.65002661
49	21.34147200	19.65129813	18.16872173	16.86275139	15.70757227
50	21.48218462	19.76200778	18.25592546	16.93151790	15.76186064

n	4%	4½%	5%	5½%	6%
51	21.61748521	19.86795003	18.33897663	16.99669943	15.81306707
52	21.74758193	19.96933017	18.41807298	17.05848287	15.86139252
53	21.87267493	20.06634466	18.49340284	17.11704538	15.90697408
54	21.99295667	20.15918149	18.56514556	17.17255486	15.94997544
55	22.10861218	20.24802057	18.63347196	17.22517048	15.99054297
56	22.21981940	20.33303404	18.69854473	17.27504311	16.02881412
57	22.32674943	20.41438664	18.76951879	17.32231575	16.06491898
58	22.42956676	20.49223602	18.81954170	17.36712393	16.09898017
59	22.52852957	20.56673303	18.87575400	17.40959614	16.13111337
60	22.62348977	20.63802204	18.92928953	17.44985416	16.16142771
61	22.71489421	20.70624118	18.98027574	17.48801343	16.19002614
62	22.80278289	20.77152266	19.02883404	17.52418334	16.21700579
63	22.88729124	20.83399298	19.07508003	17.55846762	16.24245829
64	22.96854927	20.89377319	19.11912384	17.59096457	16.26647009
65	23.04668199	20.95097913	19.16107033	17.62176737	16.28912272
66	23.12180961	21.00572165	19.20101936	17.65096433	16.31049314
67	23.19404770	21.05810684	19.23906606	17.67863917	16.33065390
68	23.26350740	21.10823621	19.27530101	17.70487125	16.34967349
69	23.33029558	21.15620690	19.30981048	17.72973579	16.36761650
70	23.39451498	21.20211187	19.34267665	17.75330406	16.38454387
71	23.45626440	21.24604007	19.37397776	17.77564366	16.40051308
72	23.51563885	21.28807662	19.40378834	17.79681864	16.41557838
73	23.57272966	21.32830298	19.43217937	17.81688970	16.42979093
74	23.62762468	21.36679711	19.45921845	17.83591441	16.44319899
75	23.68040834	21.40363360	19.48496995	17.85394731	16.45584810
76	23.73116187	21.43888383	19.50949519	17.87104010	16.46778123
77	23.77996333	21.47261611	19.53285257	17.88724180	16.47903889
78	23.82688782	21.50489579	19.55509768	17.90259887	16.48965933
79	23.87200752	21.53578545	19.57628351	17.91715532	16.49967862
80	23.91539185	21.56534493	19.59646048	17.93095291	16.50913077
81	23.95710755	21.59363151	19.61567665	17.94403120	16.51804790
82	23.99721879	21.62070001	19.63397776	17.95642767	16.52646028
83	24.03578730	21.64660288	19.65140739	17.96817789	16.53439649
84	24.07287241	21.67139032	19.66800704	17.97931554	16.54188348
85	24.10853116	21.69511035	19.68381623	17.98987255	16.54894668
86	24.14281842	21.71780895	19.69887260	17.99987919	16.55561008
87	24.17578694	21.73953009	19.71321200	18.00936416	16.56189630
88	24.20748745	21.76031588	19.72686857	18.01835466	16.56782670
89	24.23796870	21.78020658	19.73987483	18.02687645	16.57342141
90	24.26727759	21.79924075	19.75226174	18.03495398	16.57869944
91	24.29545923	21.81745526	19.76405880	18.04261041	16.58367872
92	24.32255695	21.83488542	19.77529410	18.04986769	16.58837615
93	24.34861245	21.85156499	19.78599438	18.05674662	16.59280769
94	24.37366582	21.86752631	19.79618512	18.06326694	16.59698839
95	24.39775559	21.88280030	19.80589059	18.06944734	16.60093244
96	24.42091884	21.89741655	19.81513390	18.07530553	16.60465325
97	24.44319119	21.91140340	19.82393705	18.08085832	16.60816344
98	24.46460692	21.92478794	19.83232100	18.08612164	16.61147494
99	24.48519896	21.93759612	19.84030571	18.09111055	16.61459900
100	24.50499900	21.94985274	19.84791020	18.09583939	16.61754623

The Periodic Payment That Will Amount to One.
The Sinking Fund.

$$\frac{1}{s_n} = \frac{i}{s^n - 1}$$

n	1%	1¼%	1½%	1¾%	2%
1	1.00	1.00	1.00	1.00	1.00
2	.49751244	.49689441	.49627792	.49566295	.49504950
3	.33002211	.32920117	.32838296	.32756746	.32675467
4	.26428109	.24536102	.24444478	.24353237	.24262375
5	.19603980	.19506211	.19408932	.19312142	.19215839
6	.16254837	.16153381	.16052521	.15952256	.15852581
7	.13862828	.13758872	.13655616	.13553059	.13451196
8	.12069029	.11963314	.11959402	.11754292	.11650980
9	.10674036	.10567055	.10460982	.10355813	.10251544
10	.09558208	.09450307	.09343418	.09237534	.09132653
11	.08645408	.08536839	.08429384	.08323038	.08217794
12	.07884879	.07775831	.07667999	.07561337	.07455960
13	.07241482	.07132100	.07024036	.06917283	.06811835
14	.06690177	.06580515	.06472332	.06365562	.06260197
15	.06212378	.06102646	.05994436	.05887739	.05782547
16	.05794460	.05684672	.05576508	.05469958	.05365013
17	.05425806	.05316023	.05207966	.05101623	.04996984
18	.05098205	.04988479	.04880578	.04774492	.04670210
19	.04805175	.04695548	.04587847	.04482061	.04378177
20	.04541531	.04432039	.04324574	.04219122	.04115672
21	.04303075	.04193748	.04086550	.03981464	.03878477
22	.04086372	.03977238	.03870331	.03765638	.03663140
23	.03888584	.03779666	.03673075	.03568796	.03466810
24	.03707347	.03598665	.03492410	.03388565	.03287110
25	.03540675	.03432247	.03326345	.03222952	.03122044
26	.03386888	.03278729	.03173196	.03070269	.02969923
27	.03244553	.03136677	.03031527	.02929079	.02829309
28	.03112444	.03004863	.02900108	.02798151	.02698967
29	.02989502	.02882228	.02777878	.02676424	.02577836
30	.02874811	.02767854	.02663919	.02562975	.02464992
31	.02767573	.02660942	.02557430	.02457005	.02359635
32	.02667089	.02560791	.02457710	.02357812	.02261061
33	.02572744	.02466786	.02364144	.02264779	.02168653
34	.02483997	.02378387	.02276189	.02177363	.02081867
35	.02400368	.02295111	.02193363	.02095082	.02000221
36	.02321431	.02216533	.02115240	.02017507	.01923285
37	.02246805	.02142270	.02041437	.01944257	.01850678
38	.02176150	.02071983	.01971613	.01874990	.01782057
39	.02109160	.02005365	.01905463	.01809399	.01717114
40	.02045560	.01942141	.01842710	.01747209	.01655575
41	.01985102	.01882063	.01783106	.01688170	.01597188
42	.01927563	.01824906	.01726426	.01632057	.01541729
43	.01872737	.01770466	.01672465	.01578666	.01488993
44	.01820441	.01718557	.01621038	.01527810	.01438794
45	.01770505	.01669012	.01571976	.01479321	.01390962

n	1%	1¼%	1½%	1¾%	2%
46	.01722775	.01621675	.01525125	.01433043	.01345342
47	.01677111	.01576406	.01480342	.01388836	.01301792
48	.01633384	.01533075	.01437500	.01346569	.01260184
49	.01591474	.01491563	.01396478	.01306124	.01220396
50	.01551273	.01451763	.01357168	.01267391	.01182321
51	.01512680	.01413571	.01319469	.01230269	.01145856
52	.01475603	.01376897	.01283287	.01194665	.01110909
53	.01439956	.01341653	.01248537	.01160492	.01077392
54	.01405658	.01307760	.01215138	.01127672	.01045226
55	.01372637	.01275145	.01183018	.01096129	.01014337
56	.01340824	.01243739	.01152106	.01065795	.00984656
57	.01310156	.01213478	.01122341	.01036606	.00956120
58	.01280573	.01184303	.01093661	.01008503	.00928667
59	.01252020	.01156158	.01066012	.00981430	.00902243
60	.01224445	.01128993	.01039343	.00955336	.00876797
61	.01197800	.01102758	.01013604	.00930172	.00852278
62	.01172041	.01077410	.00988751	.00905892	.00828643
63	.01147125	.01052904	.00964741	.00882455	.00805848
64	.01123013	.01029203	.00941534	.00859821	.00783855
65	.01099667	.01006268	.00919094	.00837952	.00762624
66	.01077052	.00984065	.00897386	.00816813	.00742122
67	01055136	.00962560	.00876376	.00796372	.00722316
68	.01033889	.00941724	.00856033	.00776597	.00703173
69	.01013280	.00921527	.00836329	.00757459	.00684665
70	.00993282	.00901941	.00817235	.00738930	.00666765
71	.00973870	.00882941	.00798727	.00720985	.00649446
72	.00955019	.00864501	.00780779	.00703600	.00632683
73	.00936706	.00846600	.00763368	.00686750	.00616454
74	.00918910	.00829215	.00746473	.00670413	.00600736
75	.00901609	.00812325	.00730072	.00654570	.00585508
76	.00884784	.00795910	.00714146	.00639200	.00570751
77	.00868416	.00779953	.00698676	.00624285	.00556447
78	.00852488	.00764436	.00683645	.00609806	.00542576
79	.00836983	.00749341	.00669036	.00595748	.00529123
80	.00821885	.00734652	.00654832	.00582093	.00516071
81	.00807179	.00720356	.00641019	.00568828	.00503405
82	.00792851	.00706437	.00627583	.00555936	.00491110
83	.00778887	.00692881	.00614509	.00543406	.00479173
84	.00765273	.00679675	.00601784	.00531223	.00467581
85	.00751988	.00666808	.00589396	.00519375	.00456321
86	.00739050	.00654267	.00577333	.00507850	.00445381
87	.00726418	.00642041	.00565584	.00496636	.00434750
88	.00714089	.00630119	.00554138	.00485724	.00424416
89	.00702056	.00618491	.00542984	.00475102	.00414370
90	.00690306	.00607146	.00532113	.00464760	.00404602
91	.00678832	.00596076	.00521516	.00454690	.00395101
92	.00667624	.00585272	.00511182	.00444882	.00385859
93	.00656673	.00574724	.00501104	.00435327	.00376868
94	.00645971	.00564425	.00491273	.00426017	.00368118
95	.00635511	.00554366	.00481681	.00416944	.00359602
96	.00625284	.00544541	.00472321	.00408101	.00351313
97	.00615284	.00534941	.00463186	.00399480	.00343242
98	.00605503	.00525560	.00454268	.00391074	.00335383
99	.00595936	.00516391	.00455560	.00382876	.00327729
100	.00586574	.00507428	.00437057	.00374880	.00320274

n	2¼%	2½%	2¾%	3%	3½%
1	1.00	1.00	1.00	1.00	1.00
2	.49443758	.49382716	.49321825	.49261048	.49140049
3	.32594458	.32513717	.32433243	.32353036	.32193418
4	.24171893	.24081788	.23992059	.23902705	.23725114
5	.19120021	.19024686	.18929832	.18835457	.18648137
6	.15753496	.15654997	.15557083	.15459750	.15266821
7	.13350026	.13249543	.13149747	.13050635	.12854449
8	.11548462	.11446735	.11344795	.11245639	.11047665
9	.10148170	.10045689	.09944095	.09843386	.09644601
10	.09028768	.08925876	.08823972	.08723051	.08524137
11	.08113649	.08010596	.07908629	.07807745	.07609197
12	.07351740	.07248713	.07146871	.07046209	.06848395
13	.06707686	.06604827	.06503252	.06402954	.06206157
14	.06156230	.06053653	.05952457	.05852634	.05657073
15	.05678852	.05576646	.05475917	.05376658	.05182507
16	.05261663	.05159899	.05059710	.04961085	.04768483
17	.04894039	.04792777	.04693186	.04595253	.04404313
18	.04567720	.04467008	.04368063	.04270870	.04081684
19	.04276182	.04176062	.04077802	.03981388	.03794033
20	.04014207	.03914713	.03817173	.03721571	.03536108
21	.03777572	.03678733	.03581941	.03487178	.03303659
22	.03562821	.03464661	.03368640	.03274739	.03093207
23	.03367097	.03269638	.03174410	.03081390	.02901880
24	.03188023	.03091282	.02996863	.02904742	.02727283
25	.03023599	.02927592	.02833997	.02742787	.02567404
26	.02872134	.02776875	.02684116	.02592829	.02420540
27	.02732188	.02637687	.02545776	.02456421	.02285241
28	.02602525	.02508793	.02417738	.02329323	.02160265
29	.02482081	.02389127	.02298935	.02211467	.02044538
30	.02369934	.02277764	.02188442	.02101926	.01937133
31	.02265280	.02173900	.02085453	.01999893	.01837240
32	.02167415	.02076831	.01989263	.01904662	.01744150
33	.02075722	.01985938	.01899253	.01815612	.01657242
34	.01989655	.01900675	.01814875	.01732196	.01575966
35	.01908731	.01820558	.01735645	.01653929	.01499835
36	.01832522	.01745158	.01661132	.01580379	.01428416
37	.01760643	.01674090	.01590953	.01511162	.01361325
38	.01692753	.01607012	.01524764	.01445934	.01298214
39	.01628543	.01543615	.01462256	.01384385	.01238775
40	.01567738	.01483623	.01403151	.01326238	.01182728
41	.01510087	.01426786	.01347200	.01271241	.01129822
42	.01455364	.01372876	.01294175	.01219167	.01079828
43	.01403364	.01321688	.01243871	.01169811	.01032539
44	.01353901	.01273037	.01196100	.01122985	.00987768
45	.01306805	.01226752	.01150693	.01078518	.00945343
46	.01261921	.01182676	.01107493	.01036254	.00905108
47	.01219107	.01140669	.01066358	.00996051	.00866916
48	.01178233	.01100599	.01027158	.00957777	.00830646
49	.01139179	.01062348	.00989773	.00921314	.00796167
50	.01101836	.01025806	.00954092	.00886550	.00763371

n	2¼ %	2½ %	2¾ %	3 %	3½ %
51	.01066102	.00990870	.00920014	.00853382	.00732156
52	.01031884	.00957446	.00887444	.00821718	.00702429
53	.00999094	.00925449	.00856297	.00791471	.00674100
54	.00967654	.00894799	.00826491	.00762558	.00647090
55	.00937489	.00865419	.00797953	.00734907	.00621323
56	.00908530	.00837243	.00770612	.00708447	.00596730
57	.00880712	.00810204	.00744404	.00683114	.00573245
58	.00853977	.00784244	.00719270	.00658848	.00550810
59	.00828268	.00759307	.00695153	.00635593	.00529366
60	.00803533	.00735340	.00672002	.00613296	.00508862
61	.00779724	.00712294	.00649767	.00591908	.00489249
62	.00756795	.00690126	.00628402	.00571385	.00470480
63	.00734704	.00668790	.00607866	.00551682	.00452513
64	.00713411	.00648249	.00588118	.00532760	.00435308
65	.00692878	.00628463	.00569120	.00514581	.00418826
66	.00673070	.00609398	.00550837	.00497110	.00403031
67	.00653955	.00591021	.00533236	.00480313	.00387892
68	.00635500	.00573300	.00516285	.00464159	.00373375
69	.00617677	.00556206	.00499955	.00448618	.00359453
70	.00600458	.00539712	.00484218	.00433663	.00346095
71	.00583816	.00523790	.00469048	.00419266	.00333277
72	.00567728	.00508417	.00454420	.00405404	.00320973
73	.00552169	.00493568	.00440311	.00392053	.00309160
74	.00537118	.00479222	.00426698	.00379191	.00297816
75	.00522554	.00465358	.00413560	.00366796	.00286919
76	.00508457	.00451956	.00400878	.00354849	.00276450
77	.00494808	.00438997	.00388633	.00343331	.00266390
78	.00481589	.00426463	.00376806	.00332224	.00256721
79	.00468784	.00414338	.00365382	.00321510	.00247426
80	.00456376	.00402605	.00354342	.00311175	.00238489
81	.00444350	.00391248	.00343674	.00301201	.00229894
82	.00432692	.00380254	.00333361	.00291576	.00221628
83	.00421387	.00369608	.00323389	.00282284	.00213676
84	.00410423	.00359298	.00313747	.00273313	.00206025
85	.00399787	.00349310	.00304420	.00264650	.00198662
86	.00389467	.00339633	.00295397	.00256284	.00191576
87	.00379452	.00330255	.00286667	.00248202	.00184756
88	.00369730	.00321165	.00278219	.00240393	.00178190
89	.00360291	.00312353	.00270041	.00232848	.00171868
90	.00351126	.00303809	.00262125	.00225556	.00165781
91	.00342224	.00295523	.00254460	.00218508	.00159919
92	.00333577	.00287486	.00247038	.00211694	.00154273
93	.00325176	.00279690	.00239850	.00205107	.00148834
94	.00317012	.00272126	.00232887	.00198737	.00143594
95	.00309078	.00264786	.00226141	.00192577	.00138546
96	.00301366	.00257662	.00219605	.00186619	.00133682
97	.00293868	.00250747	.00213272	.00180856	.00128995
98	.00286578	.00244034	.00207134	.00175281	.00124478
99	.00279489	.00237517	.00201185	.00169886	.00120124
100	.00272594	.00231188	.00195418	.00164667	.00115927

n	4%	4½%	5%	5½%	6%
1	1.00	1.00	1.00	1.00	1.00
2	.49019608	.48899756	.48780488	.48661800	.48543689
3	.32034854	.31877336	.31720856	.31565407	.31410981
4	.23549005	.23374365	.23201183	.23029449	.22859149
5	.18462711	.18279164	.18097480	.17917644	.17739640
6	.15076190	.14887839	.14701747	.14517895	.14336263
7	.12660961	.12470147	.12281982	.12096442	.11913502
8	.10852783	.10660965	.10472181	.10286401	.10103594
9	.09449299	.09257447	.09069008	.08883946	.08702224
10	.08329094	.08137882	.07950458	.07766777	.07586796
11	.07414904	.07224818	.07038889	.06857065	.06689294
12	.06655217	.06466619	.06282541	.06102923	.05927703
13	.06014373	.05827535	.05645577	.05468426	.05296011
14	.05466897	.05282032	.05102397	.04927912	.04758491
15	.04994110	.04811381	.04634229	.04462560	.04296176
16	.04582000	.04401537	.04226991	.04058254	.03895214
17	.04219852	.04041758	.03869914	.03704197	.03544480
18	.03899333	.03723690	.03554622	.03391992	.03235654
19	.03613862	.03440734	.03274501	.03115006	.02962086
20	.03358175	.03187614	.03024259	.02867933	.02718456
21	.03128011	.02960057	.02799611	.02646478	.02500455
22	.02919881	.02754546	.02597051	.02447123	.02304557
23	.02730906	.02568249	.02413682	.02266965	.02127848
24	.02558683	.02398703	.02247090	.02103580	.01967900
25	.02401196	.02243903	.02095246	.01954935	.01822672
26	.02256738	.02102137	.01956432	.01819307	.01690435
27	.02123854	.01971946	.01829186	.01695228	.01569717
28	.02001298	.01852081	.01712253	.01581440	.01459255
29	.01887993	.01741461	.01604551	.01476857	.01357961
30	.01783010	.01639154	.01505144	.01380539	.01264891
31	.01685535	.01544345	.01413212	.01291665	.01179222
32	.01594859	.01456320	.01328042	.01209519	.01100234
33	.01510357	.01374453	.01249004	.01133469	.01027293
34	.01431477	.01298191	.01175545	.01062958	.00959843
35	.01357732	.01227045	.01107171	.00997493	.00897386
36	.01288688	.01160578	.01043446	.00936635	.00839483
37	.01223957	.01098402	.00983979	.00879993	.00785743
38	.01163192	.01040169	.00928423	.00827217	.00735812
39	.01106083	.00985567	.00876462	.00777991	.00689377
40	.01052349	.00934315	.00827816	.00732034	.00646154
41	.01001738	.00886158	.00782229	.00689090	.00605886
42	.00954020	.00840868	.00739471	.00648927	.00568342
43	.00908989	.00798235	.00699333	.00611337	.00533312
44	.00966454	.00758071	.00661625	.00576128	.00500606
45	.00826246	.00720202	.00626173	.00543127	.00470050
46	.00788205	.00684471	.00592820	.00512175	.00441485
47	.00752189	.00650734	.00561421	.00483129	.00414768
48	.00718065	.00618858	.00531843	.00455854	.00389766
49	.00685712	.00588722	.00503965	.00430230	.00366356
50	.00655020	.00560215	.00477674	.00406145	.00344429

n	4%	4½%	5%	5½%	6%
51	.00625885	.00533232	.00452867	.00383495	.00323880
52	.00598212	.00507679	.00429450	.00362186	.00304616
53	.00571914	.00483469	.00407334	.00342130	.00286551
54	.00546912	.00460519	.00386438	.00323245	.00269602
55	.00523124	.00438754	.00366686	.00305458	.00253696
56	.00500487	.00418105	.00348010	.00288698	.00238765
57	.00478932	.00398506	.00333032	.00272900	.00224744
58	.00458401	.00379897	.00313626	.00258006	.00211574
59	.00438836	.00362221	.00297802	.00243959	.00199200
60	.00420185	.00345426	.00282818	.00230707	.00187572
61	.00402398	.00329461	.00268627	.00218202	.00176642
62	.00385430	.00314284	.00255183	.00206400	.00166366
63	.00369237	.00299848	.00242442	.00195258	.00156703
64	.00353779	.00286115	.00230365	.00184737	.00147615
65	.00339019	.00273047	.00218915	.00174800	.00139066
66	.00324921	.00260608	.00208057	.00165413	.00131022
67	.00311451	.00248765	.00197757	.00156544	.00123453
68	.00298578	.00237492	.00187986	.00148163	.00116330
69	.00286272	.00226745	.00178715	.00140242	.00109625
70	.00274506	.00216511	.00169915	.00132754	.00103313
71	.00263254	.00206759	.00161563	.00125675	.00097370
72	.00252489	.00197465	.00153633	.00118982	.00091774
73	.00242190	.00188606	.00146103	.00112652	.00086504
74	.00232334	.00180159	.00138951	.00106665	.00081542
75	.00222900	.00172104	.00132161	.00101002	.00076867
76	.00213869	.00164422	.00125709	.00095645	.00072463
77	.00205221	.00157094	.00119580	.00090577	.00086315
78	.00196939	.00150101	.00113756	.00085781	.00064407
79	.00189006	.00143434	.00108222	.00081243	.00060724
80	.00181408	.00137069	.00102962	.00076948	.00057254
81	.00174127	.00130995	.00097963	.00072884	.00053984
82	.00167150	.00125196	.00093211	.00069036	.00050902
83	.00160463	.00119663	.00088694	.00065395	.00047998
84	.00154054	.00114379	.00084399	.00061947	.00045261
85	.00147909	.00109334	.00080316	.00058683	.00042681
86	.00142018	.00104516	.00076433	.00055593	.00040241
87	.00136370	.00099915	.00072740	.00052667	.00073956
88	.00130953	.00095522	.00069228	.00049896	.00035795
89	.00125758	.00091235	.00065888	.00047273	.00033757
90	.00120775	.00087316	.00062711	.00044788	.00031836
91	.00115995	.00083486	.00059689	.00042435	.00030025
92	.00111410	.00079827	.00056815	.00040207	.00028318
93	.00107012	.00076331	.00054080	.00038096	.00026708
94	.00102789	.00072991	.00051478	.00036097	.00025189
95	.00098738	.00069799	.00049003	.00034204	.00023758
96	.00094850	.00066749	.00046648	.00032410	.00024408
97	.00091119	.00063834	.00044407	.00030711	.00021135
98	.00087538	.00061048	.00042274	.00029101	.00019935
99	.00084100	.00058385	.00040245	.00027577	.00018803
100	.00080800	.00055839	.00038314	.00026132	.00017736

TABLE VI

Effective Rate Factors.

The following are the effective rate factors for a single unit. The formula is $j_p = p(\sqrt[p]{1+i} - 1)$

p	1%	1¼%	1½%	1¾%	2%
2	.00998756	.01246118	.01494417	.01742410	.01990099
4	.00996272	.01244183	.01491636	.01739631	.01985173
12	.00995446	.01242895	.01489785	.01736119	.01981898

p	2¼%	2½%	2¾%	3%	3½%
2	.02237484	.02484567	.02731349	.02977831	.03469859
4	.02231261	.02476899	.02722087	.02966829	.03454978
12	.02227125	.02471804	.02715936	.02959524	.03445078

p	4%	4½%	5%	5½%	6%
2	.03960781	.04450483	.04939015	.05426386	.05921603
4	.03941363	.04425996	.04908894	.05390070	.05869538
12	.03928488	.04409771	.04888949	.05366039	.05841061

The effective rate factors for use with annuity tables are seldom of use to the accountant. The factors are indicated by the fraction $\frac{i}{j_p}$, as explained on pages 63 to 65, where these factors were discussed. In cases where there is an effective rate and the yearly rate is in the tables, these rules will apply:

In case of s_n or a_n: multiply the table value by the yearly rate and divide by th correct value of j_p.

In case of $\frac{1}{s_n}$ or $\frac{1}{a_n}$ multiply by the correct value of j_p and divide by the yearly rate.

TABLE VII

TEN-PLACE LOGARITHMS OF THE INTEREST RATIOS

Rate %	1 + i	log.	Rate %	1 + i	log.
			1 1/40	1.01025	.0044288591
			1/20	1.0105	.0045363179
			3/40	1.01075	.0046437500
			1/10	1.011	.0047511556
1/8	1.00125	.0005425291	1/8	1.01125	.0048585346
3/20	1.0015	.0006509536	3/20	1.0115	.0049658871
7/40	1.00175	.0007593511	7/40	1.01175	.0050732131
1/5	1.002	.0008677215	1/5	1.012	.0051805125
9/40	1.00225	.0009760649	9/40	1.01225	.0052877854
1/4	1.0025	.0010843813	1/4	1.0125	.0053950319
11/40	1.00275	.0011926707	11/40	1.01275	.0055022519
3/10	1.003	.0013009330	3/10	1.013	.0056094454
13/40	1.00325	.0014091684	13/40	1.01325	.0057166124
7/20	1.0035	.0015173768	7/20	1.0135	.0058237530
3/8	1.00375	.0016255583	3/8	1.01375	.0059308672
2/5	1.004	.0017337128	2/5	1.014	.0060379550
17/40	1.00425	.0018418404	17/40	1.01425	.0061450164
9/20	1.0045	.0019499411	9/20	1.0145	.0062520514
19/40	1.00475	.0020580149	19/40	1.01475	.0063590600
1/2	1.005	.0021660618	1/2	1.015	.0064660422
21/40	1.00525	.0022740818	21/40	1.01525	.0065729982
11/20	1.0055	.0023820749	11/20	1.0055	.0066799277
23/40	1.00575	.0024900412	23/40	1.01575	.0067868310
3/5	1.006	.0025979807	3/5	1.016	.0068937079
5/8	1.00625	.0027058934	5/8	1.01625	.0070005586
13/20	1.0065	.0028137792	13/20	1.0165	.0071073830
27/40	1.00675	.0029216383	27/40	1.01675	.0072141811
7/10	1.007	.0030294706	7/10	1.017	.0073209529
29/40	1.00725	.0031372761	29/40	1.01725	.0074276985
3/4	1.0075	.0032450548	3/4	1.0175	.0075344179
31/40	1.00775	.0033528068	31/40	1.01775	.0076411111
4/5	1.008	.0034605321	4/5	1.018	.0077477780
33/40	1.00825	.0035682307	33/40	1.01825	.0078544188
17/20	1.0085	.0036759025	17/20	1.0185	.0079610333
7/8	1.00875	.0037835477	7/8	1.01875	.0080676217
9/10	1.009	.0038911662	9/10	1.019	.0081741840
37/40	1.00925	.0039987581	37/40	1.01925	.0082807201
19/20	1.0095	.0041063233	19/20	1.0195	.0083872301
39/40	1.00975	.0042138618	39/40	1.01975	.0084937140
1	1.01	.0043213738	2	1.02	.0086001718

TABLE VII—(Continued)

TEN-PLACE LOGARITHMS OF THE INTEREST RATIOS—Continued

Rate %	1 + i	log.	Rate %	1 + i	log.
2 1.40	1.02025	.0087066034	3 1/20	1.0305	.0130479961
1/20	1.0205	.0088130091	1/10	1.031	.0132586653
3/40	1.02075	.0089193886	3/20	1.0315	.0134692323
1/10	1.021	.0090257421	1/5	1.032	.0136796973
1/8	1.02125	.0091320695	1/4	1.0325	.0138900603
3/20	1.0215	.0092383710	3/10	1.033	.0141003215
7/40	1.02175	.0093446464	7/20	1.0335	.0143104810
1/5	1.022	.0094508958	2/5	1.034	.0145205388
9/40	1.02225	.0095571192	9/20	1.0345	.0147304950
1/4	1.0225	.0096633167	1/2	1.035	.0149403498
11/40	1.02275	.0097694882			
3/10	1.023	.0098756337	11/20	1.0355	.0151501032
13/40	1.02325	.0099817533	3/5	1.036	.0153597554
7/20	1.0235	.0100878470	3/4	1.0375	.0159881054
3/8	1.02375	.0101939148	4/5	1.038	.0161973535
2/5	1.024	.0102999566	9/10	1.039	.0166155476
17/40	1.02425	.0104059726			
9/20	1.0245	.0105119627	4	1.04	.0170333393
19/40	1.02475	.0106179270	1/10	1.041	.0174507295
1/2	1.025	.0107238654	1/4	1.0425	.0180760636
			3/10	1.043	.0182843084
21/40	1.02525	.0108297780	2/5	1.044	.0187004987
11/20	1.0255	.0109356647	1/2	1.045	.0191162904
23/40	1.02575	.0110415256			
3/5	1.026	.0111473608	3/5	1.046	.0195316845
5/8	1.02625	.0112531701	3/4	1.0475	.0201540316
13/20	1.0265	.0113589537	4/5	1.048	.0203612826
27/40	1.02675	.0114647115	9/10	1.049	.0207754882
7/10	1.027	.0115704436			
29/40	1.02725	.0116761499	5	1.05	0211892991
3/4	1.0275	.0117818305	1/2	1.055	.0232524596
31/40	1.02775	.0118874854			
4/5	1.028	.0119931147	6	1.06	.0253058653
33/40	1.02825	.0120987182	1/2	1.065	.0273496078
17/20	1.0285	.0122042960			
7/8	1.02875	.0123098482	7	1.07	.0293837777
9/10	1.029	.0124153748	1/2	1.075	.0314084643
37/40	1.02925	.0125208757			
19/20	1.0295	.0126263510	8	1.08	.0334237555
39/40	1.02975	.0127318006			
3	1.03	.0128372247	9	1.09	.0374264979
			10	1.10	.0413926852